Minimum Coverings of K_n with Hexagons

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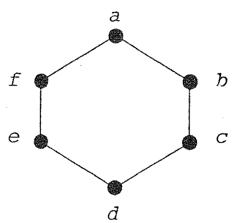
Abstract

The edge set of K_n cannot be decomposed into edge-disjoint hexagons (or 6-cycles) when $n \not\equiv 1$ or $9 \pmod{12}$. We discuss adding edges to the edge set of K_n so that the resulting graph can be decomposed into edge-disjoint hexagons. This paper gives the solution to this minimum covering of K_n with hexagons problem.

1 Introduction

A *hexagon system* is a pair (S, H) where H is a collection of edge-disjoint hexagons which partition the edge set of the complete undirected graph K_n with vertex set S. The number $|S| = n$ is called the *order* of the hexagon system, and it is easily seen that $|H| = n(n-1)/12$. In what follows we will denote the hexagon



by any cyclic shift of (a, b, c, d, e, f) or (a, f, e, d, c, b).

Example 1.1. (hexagon systems of orders 9 and 13)
S_1 ={1,2,3,4,5,6,7,8,9}; H_1 = {(1,2,3,6,7,8), (3,4,5,6,8,9), (1,3,7,4,6,9), (2,4,1,5,3,8), (2,9,4,8,5,7), (1,6,2,5,9,7)}
S_2 = {1,2,3,4,5,6,7,8,9,10,11,12,13}; H_2 = {(1,2,4,7,3,8), (13,1,3,6,2,7), (2,3,5,8,4,9), (12,13,2,5,1,6), (11,12,1,4,13,5), (10,11,13,3,12,4), (9,10,12,2,11,3), (8,9,11,1,10,2), (7,8,10,13,9,1), (6,7,9,12,8,13), (5,6,8,11,7,12), (4,5,7,10,6,11), (3,4,6,9,5,10)}

It is well-known that the *spectrum* for hexagon systems (i.e., the set of all n for which a hexagon system of order n exists) is *precisely* the set of all $n \equiv 1$ or $9 \pmod{12}$. (See [1, 5] for example.)

If $n \not\equiv 1$ or $9 \pmod{12}$, there is not a hexagon system of order n. Some natural questions arise for such n. One such question is: "What is the maximum number of edge-disjoint hexagons that can be removed from the edge set of K_n and what do the remaining edges that are not used in the hexagons look like?" This "maximum packing" problem is settled in [2, 3].

Another question to consider is: "What is the fewest number of edges that need to be added to the edge set of K_n so that the edges of the resulting graph can be decomposed into edge-disjoint hexagons, and what does the collection of added edges look like?" This question will be answered in this paper, but first we need a few definitions. A *covering* of K_n with hexagons is a pair (S, C) where C is a collection of edge-disjoint hexagons which partition the edges of $K_n \cup P$ where $P \subset E(\lambda K_n)$. The collection of edges belonging to P is called the *padding* and, as with hexagon systems, n is called the *order* of the covering. If $|P|$ is as small as possible, the covering is called a *minimum covering*. A covering is *simple* if $\lambda = 1$, i.e., the padding P is a simple graph. Since a hexagon system is a decomposition of the edges of K_n with no edges added, it is a minimum covering with padding the empty set. Throughout this paper we will refer to minimum coverings of K_n with hexagons simply as minimum coverings.

2 Necessary Conditions for Minimum Coverings

We will begin with necessary conditions for simple minimum coverings, and then expand on these conditions for minimum coverings with $\lambda > 1$.

Simple Minimum Coverings

If n is odd, every vertex of K_n has even degree, and since each vertex in a hexagon is incident with two edges in that hexagon, we know that each vertex of the padding must have even degree so that each vertex of $K_n \cup P$ has even degree. As we have noted, if $n \equiv 1$ or $9 \pmod{12}$ a hexagon system of order n exists, and this is a minimum covering with padding the empty set. If $n \equiv 3$ or $7 \pmod{12}$, $6|(\binom{n}{2} + 3)$, hence the smallest possible padding would have three edges, and each vertex having even degree implies the padding would be a 3-cycle. If $n \equiv 11 \pmod{12}, 6|(\binom{n}{2} + 5)$, so the smallest possible padding would have five edges, with each vertex having even degree, and the only such simple graph is a 5-cycle. If $n \equiv 5 \pmod{12}, 6|(\binom{n}{2} + 2)$, but there is no simple graph with 2 edges and each vertex having even degree, so the smallest possible simple paddings would each have 8 edges with each vertex having even degree. There are 7 such graphs, as we shall see in the next section.

In order for $K_n \cup P$ to have even degree when n is even, P must be a spanning subgraph of K_n with each vertex having odd degree. If $n \equiv 0$ or $6 \pmod{12}, 6|(\binom{n}{2} + \frac{n}{2})$, hence the smallest possible padding is 1-factor, which is the smallest spanning subgraph of odd degree. For $n \equiv 2, 4, 8$ or $10 \pmod{12}$, $6|(\binom{n}{2} + \frac{n}{2} + 4)$, so the smallest possible paddings would each have $\frac{n}{2}+4$ edges. There are several such spanning subgraphs of odd degree! If we have a padding with $\frac{n}{2}+4$ edges, the sum of the degrees of its vertices is $n+8$. Since each vertex must have odd degree, the only possible degree sequences for the paddings are (9,1,1,...,1), (7,3,1,1,...,1), (5,5,1,1,...1), (5,3,3,1,1,...1), and (3,3,3,3,1,1,...,1).

Minimum Coverings for $\lambda > 1$

Allowing $\lambda > 1$ will reduce the number of edges in the padding in only one case, namely $n \equiv 5 \pmod{12}$. As mentioned before, for such n, $6|(\binom{n}{2}+2)$. Also, each vertex of the padding must have even degree, so for $\lambda > 1$ the padding for a minimum covering of order $n \equiv 5 \pmod{12} \geq 17$ is a double-edge.

Also, there are cases for which allowing $\lambda > 1$ does not change the possible padding for a minimum covering. For $n \equiv 0$ or $6 \pmod{12}$ the padding is a 1-factor for all λ, and for $n \equiv 3$ or $7 (\bmod 12)$, the padding is a 3-cycle for all λ.

For $n \equiv 11 \pmod{12}$ there are 3 more possible paddings for $\lambda > 1$, each having five edges with each vertex having even degree. If $n \equiv 2, 4, 8$ or $10 \pmod{12}$, allowing $\lambda > 1$ gives several more possible paddings in each congruency class, each being an odd degree spanning subgraph of λK_n with $\frac{n}{2}+4$ edges.

3 Small Cases

We begin this section with an example of a minimum covering, and then provide a table with all of the possible paddings for each value of n and λ. The minimum coverings are available from the author on request.

Example 3.1. (K_7,C), $\lambda = 1$: P = {(1,2), (2,3), (1,3)};
C = {(1,2,3,4,6,7), (1,3,2,5,7,4), (1,2,7,3,5,6), (1,3,6,2,4,5)}

Table 3.1: Paddings for Minimum Coverings

n	λ	Padding
7	1	{(1,2), (2,3), (1,3)}
15	1	{(1,2), (2,3), (1,3)}
17	1	{(1,2), (2,3), (3,4), (4,5), (5,6), (6,7), (7,8), (1,8)}
17	1	{(5,6), (6,7), (7,8), (11,12), (9,10), (10,11), (5,8), (9,12)}
17	1	{(1,2), (2,3), (3,4), (4,5), (5,1), (5,6), (6,7), (5,7)}
17	1	{(1,2), (2,3), (3,4), (4,1), (4,5), (5,6), (6,7), (7,4)}
17	1	{(1,4), (4,2), (2,3), (3,1), (4,5), (3,5), (4,6), (3,6)}
17	1	{(1,2), (2,3), (1,3), (4,5), (5,6), (6,7), (7,8), (4,8)}
17	1	{(1,2), (2,3), (3,4), (4,5), (1,5), (5,6), (3,6), (3,5)}
17	2	{(1,2), (1,2)}

Table 3.1 (Continued)

n	λ	Padding
11	2	{(1,2), (2,3), (3,4), (4,5), (1,5)}
11	2	{(1,2), (2,3), (1,3), (3,4), (3,4)}
11	2	{(1,2), (1,3), (2,3), (4,5), (4,5)}
11	3	{(1,3), (2,3), (1,2), (1,2), (1,2)}
6	1	{(1,5), (2,3), (4,6)}
8	1	{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (5,6), (7,8)}
8	1	{(1,3), (3,5), (5,7), (3,4), (5,6), (2,4), (4,6), (6,8)}
8	1	{(1,2), (2,3), (3,5), (2,5), (3,4), (5,6), (6,7), (6,8)}
8	1	{(1,2), (2,3), (3,4), (4,5), (2,5), (3,5), (4,6), (7,8)}
8	1	{(1,4), (2,4), (3,4), (4,5), (4,7), (5,7), (5,6), (7,8)}
8	2	{(1,2), (1,3), (1,3), (3,4), (5,6), (5,6), (5,7), (6,8)}
8	2	{(2,4), (3,4), (1,4), (4,5), (4,5), (5,6), (5,7), (5,8)}
8	2	{(1,2), (2,3), (2,3), (3,4), (4,5), (4,6), (6,7), (6,8)}
8	2	{(1,2), (2,3), (2,3), (3,4), (3,5), (3,6), (6,7), (6,8)}
8	2	{(1,2), (1,2), (1,3), (2,4), (3,5), (3,6), (4,7), (4,8)}
8	2	{(1,2), (1,3), (1,4), (1,5), (1,5), (5,6), (6,7), (6,8)}
8	2	{(1,2), (2,3), (2,3), (3,4), (4,5), (4,6), (4,7), (4,8)}
8	2	{(1,2), (2,3), (2,4), (3,4), (3,4), (4,5), (4,6), (7,8)}
8	2	{(1,2), (2,3), (2,4), (3,4), (3,4), (5,6), (5,7), (5,8)}
8	2	{(1,2), (1,3), (1,4), (1,5), (1,6), (5,6), (5,6), (7,8)}
8	2	{(1,2), (1,3), (1,3), (2,4), (2,4), (3,4), (5,6), (7,8)}
8	2	{(1,2), (1,3), (1,3), (3,4), (3,5), (3,5), (5,6), (7,8)}
8	2	{(1,2), (2,3), (2,3), (3,4), (4,5), (4,5), (5,6), (7,8)}
8	2	{(1,2), (1,3), (1,3), (2,3), (2,3), (3,4), (5,6), (7,8)}
8	3	{(1,2), (2,3), (2,3), (3,4), (3,4), (3,4), (5,6), (7,8)}
8	3	{(1,2), (1,2), (1,2), (2,3), (2,4), (4,5), (4,6), (7,8)}
8	3	{(1,3), (2,3), (3,4), (3,4), (3,4), (4,5), (4,6), (7,8)}
8	3	{(1,2), (1,2), (1,2), (2,3), (2,4), (5,6), (5,7), (5,8)}
8	3	{(1,2), (1,2), (1,2), (3,4), (3,4), (3,4), (5,6), (7,8)}
8	3	{(1,2), (1,2), (1,2), (3,4), (3,5), (3,5), (5,6), (7,8)}
8	3	{(1,2), (1,2), (1,2), (3,4), (3,5), (3,6), (3,7), (3,8)}
8	3	{(1,2), (1,2), (1,2), (3,4), (3,5), (3,6), (6,7), (6,8)}
8	4	{(1,2), (1,2), (1,2), (1,2), (1,3), (2,4), (5,6), (7,8)}
8	5	{(1,2), (1,2), (1,2), (1,2), (1,2), (3,4), (5,6), (7,8)}
14	1	{(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(1,9),(1,10),(11,12),(13,14)}
14	1	{(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(8,9),(8,10),(11,12),(13,14)}
14	1	{(1,2),(3,6),(4,5),(2,3),(2,4),(3,4),(7,8),(8,9),(8,10),(11,12),(13,14)}
14	1	{(1,2),(1,3),(1,4),(1,5),(1,6),(6,7),(6,8),(6,9),(6,10),(11,12),(13,14)}
14	1	{(1,2),(1,3),(1,4),(1,5),(1,6),(5,7),(5,8),(6,9),(6,10),(11,12),(13,14)}

Table 3.1 (Continued)

n	λ	Padding
14	1	{(1,2),(1,3),(1,4),(1,5),(1,6),(6,7),(6,8),(8,9),(8,10),(11,12),(13,14)}
14	1	{(1,2),(1,3),(1,4),(2,5),(2,6),(3,7),(3,8),(4,9),(4,10),(11,12),(13,14)}
14	1	{(1,3),(2,3),(3,4),(4,5),(4,6),(6,7),(6,8),(8,9),(8,10),(11,12),(13,14)}
14	1	{(1,3),(2,3),(3,4),(4,5),(4,6),(7,9),(8,9),(9,10),(10,11),(10,12),(13,14)}
14	1	{(1,3),(2,3),(3,4),(5,6),(5,7),(5,8),(6,9),(6,10),(7,11),(7,12),(13,14)}
14	1	{(1,2),(1,3),(1,4),(1,5),(1,6),(6,7),(6,8),(9,11),(10,11),(11,12),(13,14)}
14	1	{(1,2),(1,3),(1,4),(1,5),(1,6),(7,9),(8,9),(9,10),(10,11),(10,12),(13,14)}
14	1	{(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(9,11),(10,11),(11,12),(13,14)}
14	1	{(1,2),(1,3),(1,4),(1,5),(1,6),(7,8),(7,9),(7,10),(7,11),(7,12),(13,14)}
14	1	{(1,2),(1,3),(1,4),(1,5),(1,6),(7,8),(7,9),(7,10),(11,12),(11,13),(11,14)}
14	1	{(1,2),(2,3),(2,5),(4,5),(5,6),(7,8),(8,9),(8,10),(12,13),(12,14),(11,12)}
14	2	{(1,2),(2,3),(2,3),(3,4),(3,5),(3,6),(7,8),(7,9),(7,10),(11,12),(13,14)}
14	2	{(1,2),(2,3),(2,3),(3,4),(4,5),(4,6),(7,8),(7,9),(7,10),(11,12),(13,14)}
14	2	{(1,2),(2,3),(2,3),(3,4),(3,5),(3,6),(3,7),(3,8),(9,10),(11,12),(13,14)}
14	3	{(1,2),(1,2),(1,2),(3,4),(3,5),(3,6),(7,8),(7,9),(7,10),(11,12),(13,14)}
14	3	{(1,2),(1,2),(1,2),(2,3),(2,4),(2,5),(2,6),(7,8),(9,10),(11,12),(13,14)}
20	1	{(1,2), (1,3), (1,4), (5,6), (5,7), (5,8), (9,10), (9,11), (9,12),(13,14), (13,15), (13,16), (17,18), (19,20)}
10	1	{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (5,6), (7,8), (9,10)}
10	1	{(1,2), (2,3), (2,7), (7,8), (3,6), (6,7), (3,4), (5,6), (9,10)}
10	1	{(1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (1,10)}
10	1	{(1,3), (2,3), (3,4), (4,5), (4,7), (5,6), (5,7), (7,8), (9,10)}
10	1	{(1,2), (2,3), (2,6), (3,6), (3,4), (4,5), (4,6), (7,8), (9,10)}
10	1	{(1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (4,9), (4,10)}
10	1	{(1,2), (2,3), (2,5), (3,5), (5,6), (3,4), (7,9), (8,9), (9,10)}
10	1	{(1,2), (1,3), (1,4), (1,5), (1,6), (6,7), (6,8), (6,9), (6,10)}
10	1	{(1,2), (1,3), (1,4), (1,5), (1,6), (5,7), (5,8), (6,9), (6,10)}
10	1	{(1,2), (1,3), (1,4), (1,5), (1,6), (6,7), (6,8), (8,9), (8,10)}
10	1	{(1,2), (1,3), (1,4), (2,5), (2,6), (3,7), (3,8), (4,9), (4,10)}
10	1	{(1,2), (1,3), (1,4), (1,5), (1,6), (5,6), (5,8), (6,7), (9,10)}
10	1	{(1,3), (2,3), (3,4), (4,5), (4,6), (6,7), (6,8), (8,9), (8,10)}
10	2	{(1,2), (1,3), (1,3), (3,4), (5,7), (5,6), (5,6), (6,8), (9,10)}
10	2	{(1,4), (2,4), (3,4), (4,5), (4,5), (5,6), (5,7), (5,8), (9,10)}
10	2	{(1,2), (2,3), (2,3), (3,4), (4,5), (4,6), (6,7), (6,8), (9,10)}
10	2	{(1,2), (2,3), (2,3), (3,4), (3,5), (3,6), (3,7), (3,8), (9,10)}
10	2	{(1,2), (2,3), (2,3), (3,4), (3,5), (3,6), (6,7), (6,8), (9,10)}
10	2	{(1,2), (1,2), (1,3), (2,4), (3,5), (3,6), (4,7), (4,8), (9,10)}
10	2	{(1,2), (1,3), (1,4), (1,5), (1,5), (5,6), (6,7), (6,8), (9,10)}
10	2	{(1,2), (2,3), (2,3), (3,4), (4,5), (4,6), (4,7), (4,8), (9,10)}

Table 3.1 (Continued)

n	λ	Padding
10	2	{(1,2), (2,3), (2,4), (3,4), (3,4), (4,5), (4,6), (7,8), (9,10)}
10	2	{(1,2), (2,3), (2,4), (3,4), (3,4), (5,6), (5,7), (5,8), (9,10)}
10	2	{(1,2), (1,3), (1,4), (1,5), (1,6), (5,6), (5,6), (7,8), (9,10)}
10	2	{(1,2), (1,3), (1,3), (3,4), (2,4), (2,4), (5,6), (7,8), (9,10)}
10	2	{(1,2), (1,3), (1,3), (3,4), (3,5), (3,5), (5,6), (7,8), (9,10)}
10	2	{(1,2), (2,3), (2,3), (3,4), (4,5), (4,5), (5,6), (7,8), (9,10)}
10	2	{(1,2), (1,3), (1,3), (2,3), (2,3), (3,4), (5,6), (7,8), (9,10)}
10	2	{(1,2), (1,3), (1,4), (1,5), (1,5), (5,6), (7,8), (7,9), (7,10)}
10	2	{(1,2), (2,3), (2,3), (3,4), (4,5), (4,6), (7,8), (7,9), (7,10)}
10	3	{(1,2), (2,3), (2,3), (3,4), (3,4), (3,4), (5,6), (7,8), (9,10)}
10	3	{(1,2), (1,2), (1,2), (2,3), (2,4), (4,5), (4,6), (7,8), (9,10)}
10	3	{(1,2), (1,2), (1,2), (2,3), (2,4), (2,5), (2,6), (7,8), (9,10)}
10	3	{(1,3), (2,3), (3,4), (3,4), (3,4), (4,5), (4,6), (7,8), (9,10)}
10	3	{(1,2), (1,2), (1,2), (2,3), (2,4), (5,6), (5,7), (5,8), (9,10)}
10	3	{(1,2), (1,2), (1,2), (3,4), (3,4), (3,4), (5,6), (7,8), (9,10)}
10	3	{(1,2), (1,2), (1,2), (3,4), (3,5), (3,5), (5,6), (7,8), (9,10)}
10	3	{(1,2), (1,2), (1,2), (3,4), (3,5), (3,6), (3,7), (3,8), (9,10)}
10	3	{(1,2), (1,2), (1,2), (3,4), (3,5), (3,6), (6,7), (6,8), (9,10)}
10	3	{(1,2), (1,2), (1,2), (3,4), (3,5), (3,6), (7,8), (7,9), (7,10)}
10	4	{(1,2), (1,2), (1,2), (1,2), (1,3), (2,4), (5,6), (7,8), (9,10)}
10	5	{(1,2), (1,2), (1,2), (1,2), (1,2), (3,4), (5,6), (7,8), (9,10)}
16	1	{(1,3), (2,3), (3,4), (4,5), (4,6), (7,9), (8,9), (9,10), (10,11), (10,12), (13,14), (15,16)}
16	1	{(1,3), (2,3), (3,4), (5,6), (5,7), (5,8), (6,9), (6,10), (7,11), (7,12), (13,14), (15,16)}
16	1	{(1,2), (1,3), (1,4), (1,5), (1,6), (6,7), (6,8), (9,11), (10,11), (11,12), (13,14), (15,16)}
16	1	{(1,2), (1,3), (1,4), (1,5), (1,6), (7,8), (8,9), (8,10), (10,11), (10,12), (13,14), (15,16)}
16	1	{(1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (9,10), (9,11), (9,12), (13,14), (15,16)}
16	1	{(1,2), (1,3), (1,4), (1,5), (1,6), (7,8), (7,9), (7,10), (7,11), (7,12), (13,14), (15,16)}
16	1	{(1,2), (1,3), (1,4), (1,5), (1,6), (7,9), (8,9), (9,10), (11,13), (12,13), (13,14), (15,16)}
16	1	{(1,2), (2,3), (2,4), (4,5), (4,6), (7,8), (8,9), (8,10), (11,12), (12,13), (12,14), (15,16)}
16	1	{(1,2), (1,3), (1,4), (5,6), (5,7), (5,8), (9,10), (9,11), (9,12), (13,14), (13,15), (13,16)}

4 Minimum Coverings

Before we can give the constructions for minimum coverings, we need a few more definitions. A *bipartite hexagon* system is a triple(X, Y, C) where C is a collection of edge-disjoint hexagons which partition the edge set of the complete undirected bipartite graph with vertex set $X \cup Y$ where $X \cap Y = \emptyset$. If $|X| = x$ and $|Y| = y$ the bipartite hexagon system is said to have order (x, y). We also need the following corollary to Sotteau's Theorem for our constructions.

Corollary 4.1. [6] *There exists a bipartite hexagon system of order* $(6n, 2m)$ *for all* $6n \geq 6$ *and* $2m \geq 4$.

The n+12 Minimum Covering Construction for n Odd

Let (K_n, C_1) be a minimum covering of odd order $n \geq 7$ based on $X \cup \{\infty\}$, with padding P, and (K_{13}, H_1) a hexagon system of order 13 based on $Y \cup \{\infty\}$. Since n is odd, $n - 1$ is even, implying $|X|$ is even, and since $|Y| = 12$, Corollary 4.1 guarantees the existence of a $BHS(X, Y, C_2)$. Define a collection of hexagons C on $X \cup Y \cup \{\infty\}$ by $C = C_1 \cup C_2 \cup H_1$. Then (K_{n+12}, C) is a minimum covering of order $n + 12$ with padding P.

Theorem 4.2. *If* $n \equiv 3$ *or* $7 \pmod{12} \geq 7$*, there exists a minimum covering of* K_n *with padding* P *if and only if* P *is a 3-cycle.*

Proof. Beginning with minimum coverings of orders 7 and 15, the $n + 12$ Minimum Covering Construction yields a minimum covering of every order $n \equiv 3$ or $7 \pmod{12} \geq 19$. ∎

Theorem 4.3. *If* $n \equiv 5 \pmod{12} \geq 17$*, there exists a simple minimum covering of* K_n *with padding* P *if and only if* P *is one of the paddings given in Table 3.1. For* $\lambda > 1$*, there exists a minimum covering of* K_n *with padding* P *if and only if* P *is a double-edge.*

Proof. Beginning with the minimum coverings of order 17 , the $n + 12$ Minimum Covering Construction yields minimum coverings of every order $n \equiv 5 \pmod{12} \geq 29$. ∎

Theorem 4.4. *If* $n \equiv 11 \pmod{12}$*, there exists a minimum covering of* K_n *with padding* P *if and only if* P *is one of the paddings given in Table 3.1.*

Proof. Beginning with the minimum coverings of order 11, the $n + 12$ Minimum Covering Construction yields minimum coverings with all possible paddings for admissible $n \geq 23$. ∎

Now we move on to minimum coverings of even order, for which we use a slight modification of the previous construction. Again, we stress the following construction is for *even n*.

The n + 6 Minimum Covering Construction for n Even

Let (K_n, C_1) be a minimum covering of even order $n \geq 6$ based on X, with padding P_1, and (K_6, C_2) a minimum covering of order 6 based on Y, with padding P_2. Since $|X|$ is even and $|Y| = 6$, Corollary 4.1 guarantees the existence of a $BHS(X, Y, C_3)$. Define a collection of hexagons C on $X \cup Y \cup \{\infty\}$ by $C = C_1 \cup C_2 \cup C_3$, and let $P = P_1 \cup P_2$. Then (K_{n+6}, C) is a minimum covering of order $n + 6$ with padding P.

Theorem 4.5. *If $n \equiv 0$ or $6 \pmod{12} \geq 6$, there exists a minimum covering of K_n with padding P if and only if P is a 1-factor.*

Proof. Beginning with the minimum covering of order 6, the $n+6$ Minimum Covering Construction yields a minimum covering of every order $n \equiv 0$ or $6 \pmod{12} \geq 12$. ∎

Theorem 4.6. *If $n \equiv 2$ or $8 \pmod{12} \geq 8$, there exists a minimum covering of K_n with padding P if and only if P is a spanning subgraph of λK_n with $\frac{n}{2} + 4$ edges, with each vertex having odd degree. The paddings for $n = 8$ are given in Table 3.1. The paddings for $n = 14$ are those given in Table 3.1 as well as those paddings for $n = 8$ along with 3 independent edges. The paddings for $n = 20$ are the padding given in Table 3.1 as well as those paddings for minimum coverings of orders 8 and 14 along with the appropriate number of independent edges.*

Proof. Beginning with the minimum coverings of orders 8, 14, and 20, the $n + 6$ Minimum Covering Construction yields all possible minimum coverings of every order $n \equiv 2$ or $8 \pmod{12}$. ∎

Theorem 4.7. *If $n \equiv 4$ or $10 \pmod{12} \geq 10$, there exists a minimum covering of K_n with padding P if and only if P is a spanning subgraph of λK_n with $\frac{n}{2} + 4$ edges, with each vertex having odd degree. The paddings for $n = 10$ are given in Table 3.1. The paddings for $n = 16$ are those given in Table 3.1 as well as those for $n = 10$ along with 3 independent edges.*

Proof. Beginning with the minimum coverings of orders 10 and 16, the $n+6$ Minimum Covering Construction yields all possible minimum coverings of every order $n \equiv 4$ or $10 \pmod{12}$. ∎

5 Summary

The following table gives a brief summary of the results in this paper.

Table 5.1: Summary of Minimum Coverings

K_n	λ	Number of Hexagons	Padding	Paddings Possible
$n \equiv 1$ or $9 \pmod{12}$	all	$\frac{n^2-n}{12}$	$\emptyset$	1
$n \equiv 3$ or $7 \pmod{12}$	all	$\frac{n^2-n+6}{12}$	3-cycle	1
$n \equiv 5 \pmod{12}$	1	$\frac{n^2-n+4}{12}$	8 edges, all vertices have even degree	7
	≥ 2	$\frac{n^2-n+16}{12}$	double-edge	1
$n \equiv 11 \pmod{12}$	all	$\frac{n^2-n+10}{12}$	5 edges, all vertices have even degree	4
$n \equiv 0$ or $6 \pmod{12}$	all	$\frac{n^2}{12}$	1-factor	1
$n \equiv 2$ or $8 \pmod{12}$	all	$\frac{n^2+8}{12}$	spanning subgraph of odd degree with $\frac{n}{2}+4$ edges	51
$n \equiv 4$ or $10 \pmod{12}$	all	$\frac{n^2+8}{12}$	spanning subgraph of odd degree with $\frac{n}{2}+4$ edges	51

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