ON MINIMAL TRIANGLE-FREE GRAPHS WITH PRESCRIBED 1-DEFECTIVE CHROMATIC NUMBER

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Abstract: A graph is **(m,k)-colourable** if its vertices can be coloured with m colours such that the maximum degree of the subgraph induced on vertices receiving the same colour is at most k. The **k-defective chromatic number $\chi_k(G)$** of a graph G is the least positive integer m for which G is (m,k)-colourable. Let f(m,k) be the smallest order of a triangle-free graph G such that $\chi_k(G) = m$. In this paper we study the problem of determining f(m,1). We show that f(3,1) = 9 and characterize the corresponding minimal graphs. For $m \geq 4$, we present lower and upper bounds for f(m,1).

1. Introduction

All graphs considered in this paper are undirected, finite, loopless and have no multiple edges. For the most part we follow the notation of Chartrand and Lesniak [5]. For a graph G, we denote the vertex set and the edge set of G by V(G) and E(G) respectively. The complement of a graph G is denoted by $\overline{G}$. For a positive integer n, P_n is a path of order n and C_n is a cycle of order n. For a subset U of V(G), the subgraph of G induced on U is denoted by G[U] and the subgraph induced on V(G) - U is

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denoted by G - U. For a vertex u of G and a subset X of V(G) let $N_G(u)$ denote the set of neighbours of u in G and $N_X(u) = N_G(u) \cap X$. The closed neighbourhood of u is denoted by N[u]. For notational convenience we write N(u) to mean $N_G(u)$, understanding the graph G from the context.

Let F be a graph. A graph G is said to be **F-free**, if it does not contain F as an induced subgraph. A graph is said to be **triangle-free** if it is K_3-free. A subset U of V(G) is said to be **k-independent** if the maximum degree of G[U] is at most k.

A graph is **(m,k)-colourable** if its vertices can be coloured with m colours such that the subgraph induced on vertices receiving the same colour is k-independent. Note that any (m,k)-colouring of a graph G partitions the vertex set of G into m subsets $V_1, V_2, \ldots, V_m$ such that every V_i is k-independent. These sets V_i are sometimes referred to as the **colour classes**. The **k-defective chromatic number** $\chi_k(G)$ of G is the smallest positive integer m for which G is (m,k)-colourable. Note that $\chi_0(G)$ is the usual chromatic number. Clearly $\chi_k(G) \leq \left\lceil \frac{p}{k+1} \right\rceil$, where p is the order of G.

These concepts have been studied by several authors. Hopkins and Staton [13] refer to a k-independent set as a k-small set. Maddox [16,17] and Andrews and Jacobson [2] refer to the same as a k-dependent set. The k-defective chromatic number has been investigated by Achuthan et al. [1]; Frick [9]; Frick and Henning [10]; Maddox [16,17]; Hopkins and Staton [13] under the name k-partition number; Andrews and Jacobson [2] under the name k-chromatic number Cowen et al. [7] and

Archdeacon [3] obtained some interesting results concerning k-defective colourings of graphs in surfaces.

Let f(m,k) be the smallest order of a triangle-free graph G such that $\chi_k(G) = m$. The determination of f(m,0) is still an open problem (see Toft [19], Problem 29). However partial results concerning this problem have been obtained by several authors. In the following we will briefly review some of these results.

Mycielski [18] constructed an m-chromatic triangle-free graph of order $2^m - 2^{m-2} - 1$ for all $m \geq 2$. Thus $f(m,0) \leq 2^m - 2^{m-2} - 1$ for all $m \geq 2$. Chvátal [6] proved that $f(4,0) = 11$ and $f(m,0) \geq \binom{m+2}{2} - 4$, $m \geq 4$. Furthermore he has shown that there is only one triangle-free graph G such that f(4,0) = 11. These results together imply that $17 \leq f(5,0) \leq 23$. Avis [4] improved the lower bound and showed that $f(5,0) \geq 19$. Using a slight extension of Avis' method Hanson and MacGillivray [12] have shown that $f(5,0) \geq 20$. Using a computer algorithm Grinstead, Katinsky and Van Stone [11] have shown that $21 \leq f(5,0) \leq 22$. Using computer searches Jensen and Royle [14] completely settled this problem and showed that f(5,0) = 22.

In Section 2, we will prove that f(3,1) = 9 and $f(m,1) \geq m^2$, for all $m \geq 4$. Furthermore, we will determine all the triangle-free graphs of order 9 whose 1-defective chromatic number is 3. Using the structure of these graphs we will improve the bound for f(4,1) and show that $f(4,1) \geq 17$. We also provide an upper bound for f(m,1).

For notational convenience the path $u_1, u_2, \ldots, u_n$ and the cycle $u_1, u_2, \ldots, u_n, u_1$ will be denoted by $u_1 u_2 \ldots u_n$ and $u_1 u_2 \ldots u_n u_1$ respectively. In all the figures a dotted line between vertices u and v implies that the edge (u,v) belongs to the complement.

2. Main Results :

The following theorem has been obtained independently by Lovász [15] and Hopkins and Staton [13].

Theorem 1: Let G be a graph with maximum degree Δ. Then

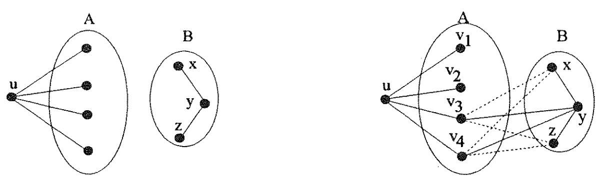
$$\chi_k(G) \leq \left\lceil \frac{\Delta+1}{k+1} \right\rceil. \qquad \square$$

We first prove two lemmas concerning triangle-free graphs.

Lemma 1 : Let G be a triangle-free graph of order 8. Then $\chi_1(G) \leq 2$.

Proof : Let u be a vertex of maximum degree in G. Let A be the set of neighbours of u in G and B = V(G) - {u} - A. Since G is triangle-free it follows that A is 0-independent.

If $|A| \geq 5$ then $|B| \leq 2$. Clearly $\chi_1(G) \leq 2$. If $|A| \leq 3$ then, by Theorem 1, $\chi_1(G) \leq 2$. Thus we will assume that $|A| = 4$. Let $\{v_1, v_2, v_3, v_4\} = A$ and $\{x, y, z\} = B$.



(a) (b)

Figure 1

If G[B] does not contain P_3 as a subgraph then $B \cup \{ u \}$ is a 1-independent set. Thus the vertices in $B \cup \{ u \}$ can be coloured with colour 1 and the vertices in A can be coloured with colour 2. Hence $\chi_1(G) \le 2$. Thus we assume that G[B] contains a path of order 3 as a subgraph. Let xyz be the P_3 in G[B] as shown in Figure 1.a.

Since $\Delta(G) = 4$, we have $|N_A(y)| \le 2$. Now if $|N_A(y)| \le 1$, clearly the sets {u, x, z} and $A \cup \{ y \}$ are both 1-independent. Thus it follows that $\chi_1(G) \le 2$ in this case. Hence we assume that $|N_A(y)| = 2$. Let v_3 and v_4 be the neighbours of y in A (see Figure 1.b). Since G is triangle-free, x and z are adjacent to neither v_3 nor v_4. Now G is (2,1)-colourable with colour classes $V_1 = \{v_1, v_2, v_3, y\}$ and $V_2 = \{u, v_4, x, z\}$. Hence $\chi_1(G) \le 2$. This proves the lemma. □

Lemma 2: Let G_i, $1 \le i \le 4$, be the graphs of order 9 shown in Figure 2. Then $\chi_1(G_i) = 3$, for $1 \le i \le 4$.

Proof: By Lemma 1, for any subgraph H of order 8 of G_i, we have $\chi_1(H) \le 2$. This implies that $\chi_1(G_i) \le 3$. Next we will show that $\chi_1(G_i) = 3$ for all i, $1 \le i \le 4$. We first prove that $\chi_1(G_1) = 3$.

Suppose $\chi_1(G_1) \le 2$. Consider a (2,1)-colouring of G_1 and let V_1, V_2 be the colour classes of G_1 such that $|V_1| \ge |V_2|$. Clearly $|V_1| \ge 5$. We will show that $z \in V_2$. Suppose $z \in V_1$. Clearly $|V_1 \cap A| \le 1$. Since V_1 is 1-independent and $G_1[B]$

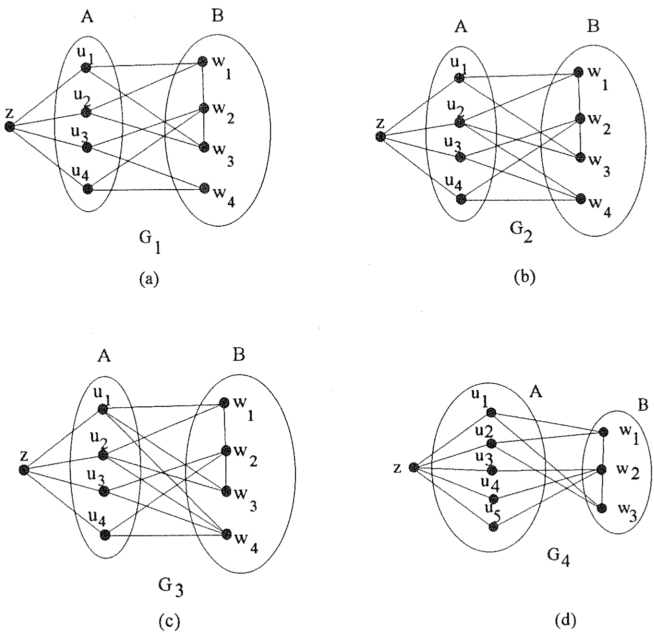


Figure 2

contains a P_3, it follows that $|V_1 \cap B| \leq 3$. Thus $5 \leq |V_1| = 1 + |V_1 \cap A| + |V_1 \cap B| \leq 5$, which implies that $|V_1 \cap A| = 1$ and $|V_1 \cap B| = 3$. Now note that every vertex of A is adjacent to two vertices of B in G. Thus V_1 is not 1-independent, a contradiction to our assumption. Hence $z \in V_2$. Now using this it is easy to show that $|V_2 \cap A| = 1$. Let $V_2 \cap A = \{u_1\}$. Clearly w_1 and $w_3 \in V_1$. Now since u_2 also belongs to V_1 it follows that V_1 is not 1-independent, a contradiction. Similarly if $V_2 \cap A = \{u_i\}$ for some i, $2 \leq i \leq 4$, we arrive at a contradiction. This proves that $\chi_1(G_1) = 3$.

We observe that G_1 is a subgraph of G_i, for $2 \le i \le 3$. This together with the fact that $\chi_1(G_i) \le 3$, for all i, gives $\chi_1(G_i) = 3$ for $2 \le i \le 3$. Now using similar arguments as in the case of G_1, it is easy to prove that $\chi_1(G_4) = 3$. This completes the proof of the lemma. □

Combining Lemmas 1 and 2 we have the following :

Theorem 2 : The smallest order of a triangle-free graph G such that $\chi_1(G) = 3$ is 9, that is, $f(3,1) = 9$. □

Theorem 3 : For any integer $m \ge 4$, $f(m,1) \ge m^2$.

Proof : Let $m \ge 3$ and G a triangle-free graph of order f(m,1) such that $\chi_1(G) = m$. Let u be a vertex of maximum degree. Since G is triangle-free, it follows that N(u) is 0-independent. Let $H \cong G - N[u]$.

Claim : $|V(H)| \ge f(m-1,1)$

Suppose $|V(H)| < f(m-1,1)$. From the definition of f(m-1,1) it follows that H is (m-2,1)-colourable. Also $\chi_1(H) = \chi_1(H \cup \{u\})$. Consider an (m-2,1)-colouring of $H \cup \{u\}$. Now by assigning a new colour to the elements of N(u) we produce an (m-1,1)-colouring of G. Thus $\chi_1(G) \le m - 1$, a contradiction to our assumption. This proves the claim.

Now $|V(G)| = f(m,1) = \Delta(G) + 1 + |V(H)|$. Using Theorem 1 and the claim established above it can be shown that

$$f(m,1) \geq 2m - 1 + f(m-1,1).$$

From the above recurrence relation it follows that

$$f(m,1) \geq (2m - 1) + (2m - 3) + \ldots + 7 + f(3,1).$$

Now incorporating the fact that $f(3,1) = 9$, we have

$$f(m,1) \geq (2m - 1) + (2m - 3) + \ldots + 7 + 9 = m^2 . \qquad \square$$

From Theorem 3 and Lemma 1 we have the following:

Remark 1: Let $m \geq 3$ be an integer. If G is a triangle-free graph of order at most $m^2 - 1$ then $\chi_1(G) \leq m - 1$. $\square$

We will now characterize triangle-free graphs of order 9 whose 1-defective chromatic number is 3.

Theorem 4: Let G be a triangle-free graph of order 9. Then $\chi_1(G) = 3$ if and only if G is isomorphic to one of the graphs of Lemma 2.

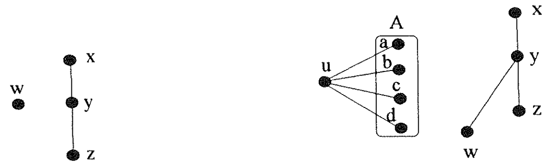
Proof : The if part follows from Lemma 2.

Let G be a triangle-free graph of order 9 with $\chi_1(G) = 3$ and u a vertex with maximum degree in G. Let A be the set of all neighbours of u. From Theorem 1 and the assumption that $\chi_1(G) = 3$ it follows that $|A| \geq 4$. Now let $H \cong G - u - A$. It can easily be shown that $\chi_1(H) = 2$. This implies that $|V(H)| \geq 3$ and hence $|A| \leq 5$.

We will divide the rest of the proof into two cases depending on the value of |A|.

Case 1 : |A| = 4

In this case |V(H)| = 4. Let A = {a,b,c,d} and V(H) = {x,y,z,w}. Since $\chi_1(H)$ = 2 , it follows that H has a P_3. Let xyz be a P_3 in H (see Figure 3.a).



(a) **(b)**

Figure 3

Now we will show that w is not adjacent to y in H. Suppose w is adjacent to y (see Figure 3.b). Since G is triangle-free, w is not adjacent to x or z. Also y is adjacent to at most one vertex of A. Therefore A ∪{y} and {u,x,z,w} are 1-independent. Thus $\chi_1(G)$ ≤ 2, a contradiction. Hence w is not adjacent to y in H. Now H is isomorphic to $P_3 \cup K_1$ or P_4 or C_4 according as w is adjacent to neither or exactly one or both of the vertices x and z.

Subcase 1.1 : H is isomorphic to $P_3 \cup K_1$

Recall that xyz is a P_3 in H. Notice that w is the isolated vertex in H (see Figure 4.a). Clearly {u,x,z,w} is 1-independent. Since $\Delta(G)$ = 4 it follows that $|N_A(y)| \le 2$. If

$|N_A(y)| \leq 1$ then $A \cup \{y\}$ is 1-independent in G. Thus $\chi_1(G) \leq 2$, a contradiction. Thus $|N_A(y)| = 2$.

Without any loss of generality let $N_A(y) = \{c,d\}$(see Figure 4.a).

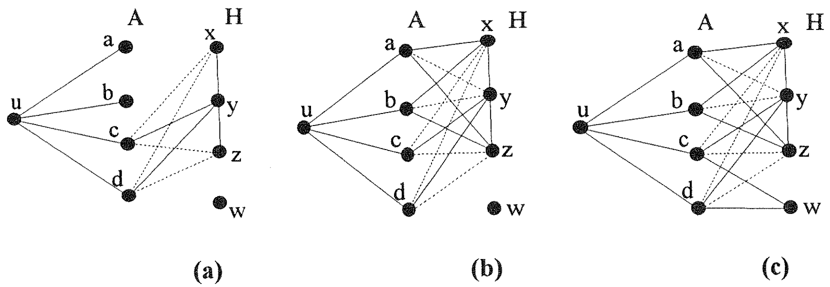


Figure 4

Consider the vertex x of H. Since G is triangle-free, (x,c) and (x,d) $\notin$ E(G). If x is adjacent to at most one of the vertices a and b then $A \cup \{x\}$ is 1-independent. Also since {u,y,z,w} is 1-independent we have $\chi_1(G) \leq 2$, a contradiction. Therefore x is adjacent to both a and b. Similarly z is not adjacent to c or d and is adjacent to both a and b (see Figure 4.b). Note that {a, b, d, y} is 1-independent. Suppose w is not adjacent to c in G. Then {u,c,x,z,w} is a 1-independent set. This implies that $\chi_1(G) \leq 2$, a contradiction. Thus w is adjacent to c. Similarly it can be shown that w is adjacent to d (see Figure 4.c). Now it is easy to see that G is isomorphic to G_1, or G_2, or G_3 according as the number of neighbours of w in {a, b} is 0 or 1 or 2.

Subcase 1.2 : H is isomorphic to P_4

Recall that xyz is a P_3 in H. We assume that w is adjacent to z in H (see Figure 5.a).

(a) **(b)**

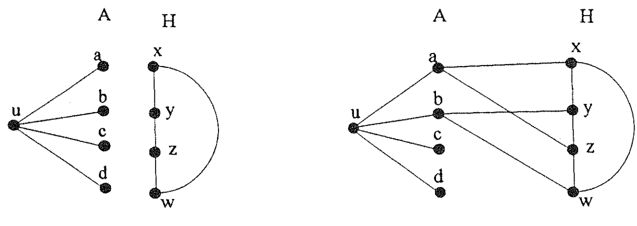
Figure 5

Since $\Delta(G) = 4$, we have $|N_A(y)| \leq 2$. Suppose $|N_A(y)| \leq 1$. Then the sets $A \cup \{y\}$ and $\{u,x,z,w\}$ form a partition of V(G) into 1-independent sets implying $\chi_1(G) \leq 2$, a contradiction to our assumption. Thus $|N_A(y)| = 2$. Similarly it can be shown that $|N_A(z)| = 2$. Since G is triangle-free, we have $N_A(y) \cap N_A(z) = \emptyset$. Without any loss of generality let us assume that $N_A(y) = \{c,d\}$ and $N_A(z) = \{a,b\}$. Again since G is triangle-free, x is not adjacent to c and d and w is not adjacent to a and b (see Figure 5.b).

It is easy to see that y is a vertex of degree 4 and the subgraph induced on V(G) - N[y] is isomorphic to $P_3 \cup K_1$ and hence we are in Subcase 1.1.

Subcase 1.3: H is isomorphic to C_4

Recall that xyz is a P_3 in H. Thus in this case w is adjacent to x and z (see Figure 6.a).



(a) (b)

Figure 6

Firstly we suppose that every vertex of H has at most one neighbour in A. If x and z do not have a common neighbour in A, then $A \cup \{x, z\}$ and $\{u, y, w\}$ form a partition of V(G) into 1-independent sets. Hence $\chi_1(G) \leq 2$, a contradiction to our assumption. Thus x and z have a common neighbour in A. Similarly it can be shown that y and w have a common neighbour in A. Without any loss of generality let a be the common neighbour of x and z and b the common neighbour of y and w (see Figure 6.b). Now it is easy to see that $\{u, b, x, z\}$ and $\{a, c, d, y, w\}$ are both 1-independent and hence $\chi_1(G) \leq 2$, a contradiction. This contradiction implies that some vertex of H has at least two neighbours in A. Without any loss of generality let $|N_A(x)| \geq 2$. Since $\Delta(G) = 4$, it follows that $|N_A(x)| = 2$. Now let $N_A(x) = \{a, b\}$ (see Figure 7.a).

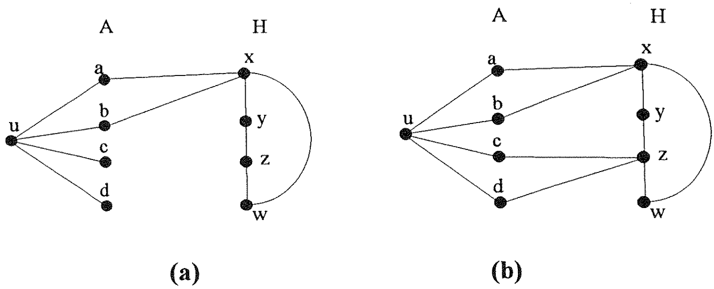
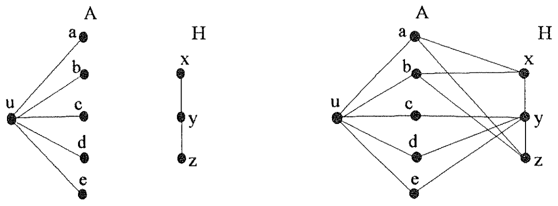


Figure 7

Now note that x is a vertex of degree 4. If the vertex z is not adjacent to both c and d then V(G) - N[x] is isomorphic to $P_3 \cup K_1$ or P_4 and hence we are in Subcase 1.1 or 1.2. Thus we assume that z is adjacent to both c and d (see Figure 7.b). Now clearly the vertices y and w do not have any neighbour in A. Thus $A \cup \{y, w\}$ and $\{u, x, z\}$ are both 1-independent and hence $\chi_1(G) \leq 2$, a contradiction. This completes the proof in Subcase 1.3.

Case 2 : $|A| = 5$

In this case $|V(H)| = 3$. Since $\chi_1(H) = 2$ and H is triangle-free, it follows that $H \cong P_3$. Let xyz be the P_3 in H and $A = \{a,b,c,d,e\}$(see Figure 8.a).



(a) (b)

Figure 8

Note that each vertex α of H has at least two neighbours in A, for otherwise $A \cup \{\alpha\}$ and $\{u\} \cup V(H) - \{\alpha\}$ provide a (2,1)-colouring of G.

Claim : $|N_A(y)| = 3$

Firstly since $\Delta(G) = 5$, $|N_A(y)| \leq 3$. If $|N_A(y)| \leq 2$ then from the above remark we have $|N_A(y)| = 2$. Without loss of generality let a and b be the neighbours of y. Clearly x and z are not adjacent to either of a and b. Thus {a,c,d,e,y} and {b,x,z,u} are both 1-independent which implies $\chi_1(G) \leq 2$, a contradiction. This proves the claim.

Without loss of generality let c,d and e be the neighbours of y. Again x and z are not adjacent to any element of {c,d,e} in G. Thus x and z have at most two neighbours in A. Combining this with the fact that any vertex of H has at least two neighbours in A we have $|N_A(x)| = |N_A(z)| = 2$. Thus $N_A(x) = N_A(z) = \{a,b\}$ (see Figure 8.b). Now it is easy to see that G is isomorphic to the graph G_4 of Lemma 2.

This completes the proof of Theorem 4. □

Theorem 5 : The smallest order of a triangle-free graph G with $\chi_1(G) = 4$ is at least 17, that is, $f(4,1) \geq 17$.

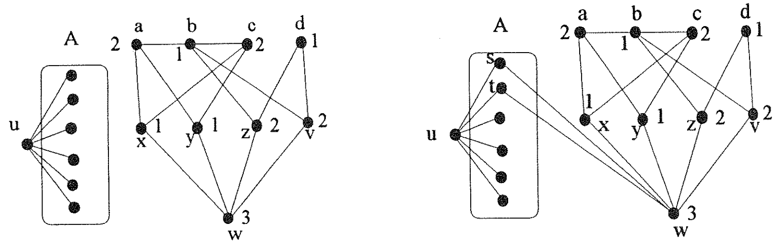
Proof : To prove the theorem, it is sufficient to show that if G is a triangle-free graph of order 16, then $\chi_1(G) \leq 3$.

Let G be a triangle-free graph of order 16. We shall prove that $\chi_1(G) \leq 3$.

Let u be a vertex of maximum degree in G and A = N(u), so $|A| = \Delta(G)$. Define $H \cong G-u-A$. It is easy to see that if $\chi_1(H) \leq 2$ then $\chi_1(G) \leq 3$. Thus we will assume that $\chi_1(H) \geq 3$. Combining this with Lemma 1 we have $|V(H)| \geq 9$. Thus $\Delta(G) = |A| \leq 6$. Now if $\Delta(G) \leq 5$, then by Theorem 1, G is (3,1)-colourable. Thus let us assume that $\Delta(G) = 6$. This implies that $|V(H)| = 9$. Applying Remark 1 with $m = 4$ to the graph H, we have $\chi_1(H) \leq 3$. Combining this with the assumption that $\chi_1(H) \geq 3$, it follows that $\chi_1(H) = 3$. Thus we have established that H is a graph of order 9 with $\chi_1(H) = 3$. From Theorem 4 it follows that H is isomorphic to one of the graphs of Lemma 2 shown in Figure 2. Let V(H) = {a,b,c,d,x,y,z,v,w}.

Firstly let us assume that H is isomorphic to G_1 of Figure 2. Consider the (3,1)-colouring of H shown in Figure 9.a.

The numbers next to the vertices a to w denote the colours assigned to the vertices. We will now extend this (3,1)-colouring of H to a (3,1)-colouring of G.



(a) **(b)**

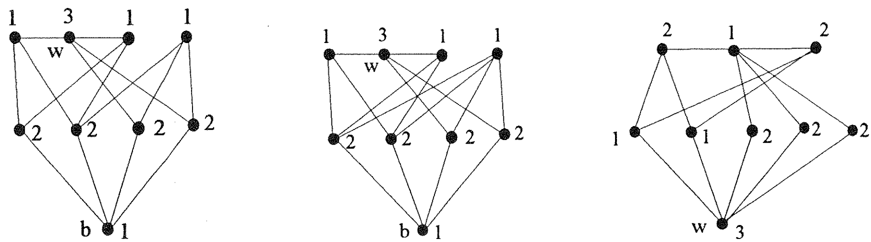
Figure 9

Observe that w is adjacent to at most two vertices of A since $\Delta(G) = 6$. If w is adjacent to at most one vertex of A then assign colour 3 to the vertices of A and assign colour 1 to u. This produces a (3,1)-colouring of G. Thus let us assume that w is joined to exactly two vertices, say, s and t of A(see Figure 9.b).

Since G is triangle-free, s and t are not adjacent to any element of {x,y,z,v}. Firstly we assign colour 3 to the elements of A - s. Now we colour s and u as follows : If s is adjacent to b, then s is not adjacent to a or c. Hence we can assign colour 2 to s and colour 1 to u. Thus we have a (3,1)-colouring of G in this case. On the other hand if s is not adjacent to b note that {s,b,d,x,y} is 1-independent and hence we assign colour 1 to s and colour 2 to u. This forms a (3,1)-colouring of G in this case. Thus when $H \cong G_1$ of Figure 2, we have extended the (3,1)-colouring of H shown in Figure 9.a to a (3,1)-colouring of G.

Now assume that H is isomorphic to G_i for some i, $2 \le i \le 4$, of Figure 2. We have reproduced those graphs in Figure 10 along with a (3,1)-colouring. In the following we will briefly explain how to extend the (3,1)-colouring of G_i to the graph G.

Firstly let i = 2 or 3. As in the case $H \cong G_1$ it is easy to produce a (3,1)-colouring of G if w has at most one neighbour in A. So we will assume that w is adjacent to exactly two vertices, say s and t of A. Colour the vertices of $A \cup \{u\}$ as follows: The vertices in A - {s} are assigned colour 3. The vertex s is assigned colour 2 or 1 according as s is or is not adjacent to the vertex b. Now the vertex u will be assigned colour 1 or 2 according as s is assigned colour 2 or 1. It is easy to check that this is a (3,1)-colouring of G.



G_2 G_3 G_4

Figure 10

Finally let $H \cong G_4$. Since $\Delta(G) = 6$, w is adjacent to at most one vertex of A. Hence we can assign colour 3 to all the elements of A and colour 1 to u. This provides a (3,1)-colouring of G and completes the proof of Theorem 5. □

Using the proof of Theorem 3 and Theorem 5 we have the following :

Corollary : For any integer $m \geq 5$, $f(m,1) \geq m^2 + 1$. □

In the following we shall prove that there exist triangle-free graphs of arbitrarily large 1-defective chromatic number. The construction is similar to the construction (of triangle-free graphs of arbitrarily large chromatic number) due to Mycielski [18].

Theorem 6 : For every positive integer n, there exists a triangle-free graph G with $\chi_1(G) = n$.

Proof : We prove Theorem 6 by induction on n. For $n = 1$ and 2 the graphs K_1 and P_3, respectively, have the required properties. Assume that H is a triangle-free graph of order p with $\chi_1(H) = k$, where $k \geq 3$. We will now construct a triangle-free graph G with $\chi_1(G) = k+1$.

Let $V(H) = \{v_1, v_2, \ldots, v_p\}$. Then define

$$V(G) = V(H) \cup \{u_i, w_i : 1 \leq i \leq p\} \cup \{x\}$$

$$E(G) = E(H) \cup E_1 \cup E_2$$

where

$$E_1 = \{(u_i,y),(w_i,y) : y \text{ is a neighbour of } v_i \text{ in } H\}$$

and

$E_2 = \{(x,u_i),(x,w_i) : 1 \le i \le p\}$.

It is easy to show that G is triangle-free. We will prove that $\chi_1(G) = k+1$. Consider a (k,1)-colouring of H which uses colours 1,2,...,k. Now assign a new colour k + 1 to all the vertices u_i and w_i, for $1 \le i \le p$, and colour 1 to the vertex x. This provides a (k+1,1)-colouring of G. Thus $\chi_1(G) \le k+1$.

To prove equality, if possible, consider a (k,1)-colouring of G, which uses colours 1,2,...,k. Without loss of generality assume that the vertex x is assigned colour 1. From this (k,1)-colouring of G we will provide a (k-1,1)-colouring of H.

Let C_α be the set of all vertices of G that are assigned colour α, $1 \le \alpha \le k$. Further, let $V_1 = C_1 \cap V(H) = \{v_1,v_2,\ldots,v_\ell\}$. Without loss of generality we suppose that for $1 \le i \le m$, the degree of v_i, in the graph $H[V_1]$ is 0 and for $m + 1 \le i \le \ell$, the degree of v_i, in the graph $H[V_1]$ is 1. The following are easily established (see Figure 11) :

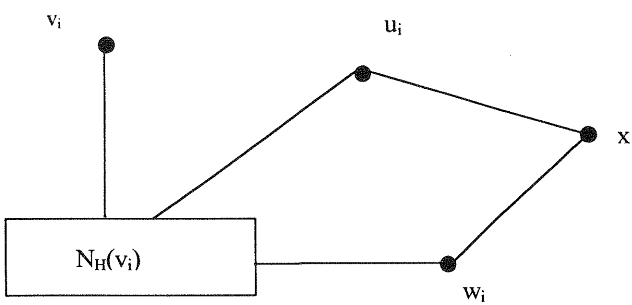


Figure 11

(i) $\left| \bigcup_{i=1}^{p} \{u_i, w_i\} \cap C_1 \right| \leq 1.$

(ii) For $1 \leq i \leq \ell$, if $u_i \in C_\alpha$ for some $\alpha \neq 1$, then

$$|C_\alpha \cap N_H(v_i)| \leq 1$$

and

$$(C_\alpha \cup \{v_i\}) \cap V(H) \text{ is 1-independent.}$$

(iii) The statement (ii) is also true for w_i, $1 \leq i \leq \ell$.

(iv) For i, $1 \leq i \leq \ell$, if $u_i, w_i \in C_\alpha$, for some $\alpha \neq 1$, then $|C_\alpha \cap N_H(v_i)| = 0$.

In the following we describe the method of changing the colour of every vertex of V_1 to some other suitable colour.

1. For $1 \leq i \leq m$, the vertex v_i is reassigned colour α, where α is such that $\{u_i, w_i\} \cap C_\alpha \neq \varnothing$.

2. Suppose $m + 1 \leq i \leq \ell$. Note that $\ell - m$ is even and $H[\{v_{m+1}, \ldots, v_\ell\}]$ is a matching. Consider v_i and v_j, $m+1 \leq i, j \leq \ell$ such that $(v_i, v_j) \in E(H)$. Clearly none of the vertices in $\{u_i, w_i, u_j, w_j\}$ is assigned colour 1, for otherwise, we have a P_3 in C_1.

   2a. If $\exists$ an $\alpha \neq 1$ such that $\{u_i, w_i, u_j, w_j\} \subseteq C_\alpha$, then both the vertices v_i and v_j are reassigned the colour α.

2b. Suppose α and β are two distinct colours such that

$\{u_i,w_i\} \cap C_\alpha \neq \emptyset$ and $\{u_j,w_j\} \cap C_\beta \neq \emptyset$. Now we

reassign the colour α to v_i and the colour β to v_j.

We repeat the steps 2a and 2b for every pair of adjacent vertices in $H[\{v_{m+1},\ldots,v_\ell\}]$.

We will now prove that this procedure results in a (k-1,1)-colouring of H. Let V_α be the set of vertices of H that have been assigned colour α, for $2 \leq \alpha \leq k$. Note that $C_\alpha \cap V(H) \subseteq V_\alpha$, for $2 \leq \alpha \leq k$. In the following, we will prove that $H[V_2]$ is 1-independent. The same arguments hold for $3 \leq \alpha \leq k$.

Suppose $H[V_2]$ is not 1-independent. Let $v_rv_sv_t$ be a P_3 in $H[V_2]$ (see Figure 12).

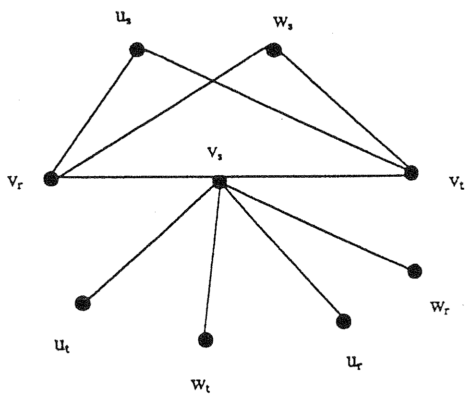


Figure 12

It is easy to see that at least one and at most two of the vertices in $\{v_r, v_s, v_t\}$ belong to C_2.

Claim : $v_s \notin C_2$, that is, v_s was originally assigned colour 1.

Suppose $v_s \in C_2$. At least one of v_r and v_t must be in C_1, say $v_r \in C_1$. Since the vertex v_r has been reassigned colour 2, from our procedure it follows that either u_r or w_r belongs to C_2, say $u_r \in C_2$. Since $u_r v_s v_t$ is a P_3 in G it follows that $v_t \notin C_2$ and hence $v_t \in C_1$. This in turn implies that either u_t or w_t belongs to C_2, say $u_t \in C_2$. But this gives a P_3 namely, $u_r v_s u_t$ in the colour class C_2 of G, a contradiction. This proves the claim.

Since the colour of v_s has been changed from 1 to 2 (by our procedure), it follows that at least one of u_s and w_s must be in C_2, say $u_s \in C_2$.

Now without loss of generality let us assume that $v_r \in C_2$. Since $v_r u_s v_t$ is a P_3 in G, it follows that $v_t \in C_1$. Since v_s and v_t are adjacent in $H[V_1]$, and they are both reassigned colour 2, it follows from our procedure that all the vertices in $\{u_s, w_s, u_t, w_t\}$ must be in C_2. But this gives a P_3, namely $u_s v_r w_s$ in C_2, a contradiction.

Thus, we have provided a (k-1,1)-colouring of H, a contradiction to the fact that $\chi_1(H) = k$. This contradiction proves that $\chi_1(G) = k+1$. This completes the proof of the theorem. □

Remark 2 : From Theorem 6 and the definition of f(m,1) it follows that, for $m \geq 4$, $f(m,1) \leq 3.\ f(m-1,1) + 1$. Now using the fact that $f(3,1) = 9$, we have

$$f(m,1) \leq 3^{m-1} + \frac{3^{m-3}-1}{2}.$$

Combining Theorem 5 and Remark 2 we have

$$17 \leq f(4,1) \leq 28.$$

Remark 3 : Theorem 6 also follows from the results of Folkman ([8], Theorem 2). However, the order of the graph constructed in Folkman's proof is larger than the order of the graph in Theorem 6.

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