On Covering Designs with Block Size 5 and Index $11 \leq \lambda \leq 21$: The case $v \equiv 0 \pmod 4$

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Let V be a finite set of order v. A (v,k,λ) covering design of index λ and block size k is a collection of k-element subsets, called blocks, such that every 2-subset of V occurs in at least λ blocks. The covering problem is to determine the minimum number of blocks, $\alpha(v,k,\lambda)$, in a covering design. It is well known that $\alpha(v,k,\lambda) \geq \lceil \frac{v}{k} \lceil \frac{v-1}{k-1} \lambda \rceil \rceil = \phi(v,k,\lambda)$, where $\lceil x \rceil$ is the smallest integer satisfying $x \leq \lceil x \rceil$. It is shown here that with the possible exception of $(v,\lambda) =$ (44, 13), (28, 17), (44, 17), $\alpha(v,5,\lambda) = \phi(v,5,\lambda) + e$ provided $v \equiv 0 \pmod 4$ and $11 \leq \lambda \leq 21$ where $e = 1$ if $\lambda(v-1) \equiv 0 \pmod 4$ and $\frac{\lambda v(v-1)}{4} \equiv -1 \pmod 5$ and e=0 otherwise.

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1. Introduction

A (v,k,λ) *covering design* (or respectively *packing design*) of order v, block size k and index λ is a collection β of k-element subsets, called blocks, of a v-set V such that every 2-subset of V occurs in at least (at most) λ blocks.

Let $\alpha(v,k,\lambda)$ denote the minimum number of blocks in a (v,k,λ) covering design; and $\sigma(v,k,\lambda)$ denote the maximum number of blocks in a (v,k,λ) packing design. A (v,k,λ) covering design with $|\beta| = \alpha(v,k,\lambda)$ is called a minimum covering design. Similarly, a (v,k,λ) packing design with $|\beta| = \sigma(v,k,\lambda)$ will be called a *maximum packing design.* It is well known that [33]

$$\alpha(v,k,\lambda) \geq \left\lceil \frac{v}{k}\left\lceil \frac{v-1}{k-1}\lambda\right\rceil\right\rceil = \phi(v,k,\lambda) \quad \text{and} \quad \sigma(v,k,\lambda) \leq \left[\frac{v}{k}\left[\frac{v-1}{k-1}\lambda\right]\right] = \psi(v,k,\lambda)$$

where $\lceil x\rceil$ is the smallest and $[x]$ is the largest integer satisfying $[x] \leq x \leq \lceil x\rceil$.

When $\alpha(v,k,\lambda) = \phi(v,k,\lambda)$ the (v,k,λ) covering design is called a *minimal covering design.* Similarly, when $\sigma(v,k,\lambda) = \psi(v,k,\lambda)$ the (v,k,λ) packing design is called an *optimal packing design.*

Many researchers have been involved in determining the covering numbers known to date (see bibliography) most notably W.H. Mills and R.C. Mullin. In one of their papers they proved the following [31].

<u>Theorem 1.1</u> Let v be an odd integer greater than 5.

(i) If $v \equiv 1 \pmod 4$ and $\lambda > 1$, then $\alpha(v,5,\lambda) = \phi(v,5,\lambda) + e$ where $e = 1$ if $\lambda(v-1) \equiv 0 \pmod 4$ and $\frac{\lambda v(v-1)}{4} \equiv -1 \pmod 5$ and $e=0$ otherwise with the exceptions that $\alpha(9,5,2) = \phi(9,5,2)+1$, $\alpha(13,5,2) = \phi(13,5,2)+1$ and the possible exceptions of the pairs $(v,\lambda) \in \{(53,2), (73,2)\}$ and,

(ii) If $v \equiv 3 \pmod 4$ and $\lambda \geq 1$ then $\alpha(v,5,\lambda) = \phi(v,5,\lambda)+e$ where e is as in (i) with the exceptions that $\alpha(15,5,\lambda) = \phi(15,5,\lambda)+1$ for $\lambda = 1, 2$ and the possible exception of the pairs $(v,\lambda) \in \{(63,2), (83,2)\}$.

In the case $v \equiv 0 \pmod 4$ and $\lambda=1$ the problem is still open.

For $v \equiv 0 \pmod 4$ and $2 \leq \lambda \leq 10$ and $\lambda = 12,16,20$ we have the following result [5] [6] [7] [8] [11] [16] [17] [23] .

<u>Theorem 1.2</u> Let $v \equiv 0 \pmod 4$ $v \geq 8$ be an integer. Then $\alpha(v,5,\lambda) = \phi(v,5,\lambda) + e$ for $2 \leq \lambda \leq 10$ and $\lambda=12,16,20$ where $e=1$ if $\lambda(v-1) \equiv 0 \pmod 4$ and $\frac{\lambda v(v-1)}{4} \equiv -1 \pmod 5$ and $e=0$ otherwise with the possible exceptions of $(v,\lambda) =$ (28,4) (24,5) (28,5) (56,5) (104,5) (124,5) (144,5) (164,5) (184,5) (28,7) (24,9)

(28,9) (56,9) (144,9) (164,9) (184,9) (224,9). In this paper we consider the remaining indices of λ where $\lambda \leq 21$ and $v \equiv 0 \pmod 4$. Specifically, we prove the following.

Theorem 1.3 Let $v \equiv 0 \pmod 4$, $v \geq 8$ be an integer. Then $\alpha(v,5,\lambda) = \phi(v,5,\lambda) + e$ for all positive integers $11 \leq \lambda \leq 21$, where e is as in theorem 1.2, with the possible exceptions of $(v,\lambda) = (44,13), (28,17), (44,17)$.

2. Recursive Constructions

In order to describe our recursive constructions we require the notions of transversal designs, group divisible designs, covering (packing) designs with a hole of size h, and balanced incomplete block designs. For the definition of these combinatorial designs see [5]. We also adopt the same notations: a $T[k,\lambda,m]$ stands for a transversal design with block size k, index λ and group size m. A $GD[k,\lambda,M,v]$ stands for a group divisible design with block size k, index λ, group sizes from M, and v is the number of points in the design. If $M = \{m\}$ then the design is denoted by $GD[k, \lambda, m, v]$. A $B[v,k,\lambda]$ stands for a balanced incomplete block design with block size k, index λ, and point set of size v. It is clear that if a $B[v, k, \lambda]$ exists then $\alpha(v, k, \lambda) = \frac{\lambda v(v-1)}{k(k-1)} = \phi(v, k, \lambda)$ and Hanani [23] has proved the following existence theorem.

Theorem 2.1 Necessary and sufficient conditions for the existence of a $B[v, 5, \lambda]$ are that $\lambda(v-1) \equiv 0 \pmod 4$ and $\lambda v(v-1) \equiv 0 \pmod{20}$ and $(v, \lambda) \neq (15, 2)$.

The following obvious lemma is most useful to us.

Lemma 2.1 If there exists a $B[v, 5, \lambda]$ and $\alpha(v, 5, \lambda') = \phi(v, 5, \lambda')$, then $\alpha(v, 5, \lambda+\lambda') = \phi(v, 5, \lambda+\lambda')$.

We also shall make use of the following [5].

Lemma 2.2 If there exists a (v, k, λ) covering design with a hole of size $h \geq k$ and $\alpha(h, k, \lambda) = \phi(h, k, \lambda)$ then $\alpha(v, k, \lambda) = \phi(v, k, \lambda)$.

In many places through this paper, instead of constructing a $(v, 5, \lambda)$ minimal covering design we construct a $(v, 5, \lambda)$ covering design with a hole of size $h \geq 5$ where $\alpha(h, 5, \lambda) = \phi(h, 5, \lambda)$ and then apply lemma 2.2

The proof of the following theorem may be found in [1], [2], [3], [18], [20], [23], [32], [34].

Theorem 2.2 There exists a $T[6, 1, m]$ for all positive integers m with the exception of $m \in \{2, 3, 4, 6\}$ and the possible exception of $m \in \{10, 14, 18, 22\}$.

Theorem 2.3 [17] If there exists a $GD[6, \lambda, 5, 5n]$ and a $(20+h, 5, \lambda)$ covering design with a hole of size h then there exists a $(20(n-1)+4u+h, 5, \lambda)$ covering design with a hole of size $4u+h$ where $0 \leq u \leq 5$.

Theorem 2.4 [17] If there exists a $GD[6, \lambda, 5, 5n]$, a $(20+h, 5, \lambda)$ covering design with a hole of size h, a $(20+h, 5, \lambda)$ minimal covering design, then there exists a $(20n+h, 5, \lambda)$ minimal covering design.

The application of the previous theorems requires the existence of a $GD[6, \lambda, 5, 5n]$. Our authority for this is the following lemma of Hanani [23, p. 286].

Lemma 2.3 There exists a $GD[6, \lambda, 5, 5n]$ for $n = 7$ and $\lambda \geq 2$.

Let k, λ, m, and v be positive integers. A modified group divisible design, $MGD[k, \lambda, m, v]$, is a quadruple $(V, \beta, \gamma, \delta)$ where V is a set of points with $|V| = v = mn$, $\gamma = \{G_1, ..., G_m\}$ is a partition of V into m sets, called groups, $\delta = \{R_1, ..., R_n\}$ is a partition of V into n sets, called rows, and β is a family of k-subsets of V, called blocks, with the following properties.

1) $|B \cap G_i| \leq 1$ for all $B \in \beta$ and $G_i \in \gamma$.
2) $|B \cap R_j| \leq 1$ for all $B \in \beta$ and $R_j \in \delta$.
3) $|G_i| = n$ for all $G_i \in \gamma$ and $|R_j| = m$ for all $R_j \in \delta$.
4) Every 2-subset $\{x, y\}$ of V such that x and y are neither in the same group nor same row is contained in exactly λ blocks.
5) $|G_i \cap R_j| = 1$ for all $G_i \in \gamma$ and $R_j \in \delta$.

A resolvable modified group divisible design, RMGD[k, λ, m, v], is a modified group divisible design the blocks of which are partitioned into parallel classes. It is clear that a RMGD[5, 1, 5, 5m] is the same as RT[5, 1, m] with one parallel class of blocks singled out, and since a RT[5, 1, m] is equivalent to a T[6, 1, m] we have the following.

Theorem 2.5 There exists a RMGD[5, 1, 5, 5m] for all positive integers m, $m \neq 2$, 3, 4, 6, with the possible exception of $m \in \{10, 14, 18, 22\}$.

The following theorem is our main recursive construction.

Theorem 2.6 [17] If there exists (1) a RMGD[5, 1, 5, 5m], (2) a GD[5, λ {4, s*}, 4m+s], where * means there is exactly one group of size s, (3) there exists a (20+h, 5, λ) covering design with a hole of size h then there exists a (20m+4u+h+s, 5, λ) covering design with a hole of size 4u+h+s where $0 \leq u \leq m-1$.

Theorem 2.7 [17] If there exists (1) a RMGD[5, 1, 5, 5m], (2) a GD[5, λ, {4, 8*}, 4m+4] where * is as before, (3) a (20, 5, λ) minimal covering design and a (24, 5, λ) covering design with a hole of size 4, then there exists a (20m+4u+4, 5, λ) covering design with a hole of size 4u+4 where $0 \leq u \leq m-1$.

Theorem 2.8 [4] If there exists (1) a RMGD[5, 1, 5, 5m], (2) a GD[5, λ, {4, s*}, 4(m−1)] and (3) a (20+h, 5, λ) covering design with a hole of size h then there exists a (24(m−1)+s+h, 5, λ) covering design with a hole of size 4(m−1)+s+h.

It is clear that the application of the above theorems requires the existence of a GD[5, 1, {4,s*}, 4m+s]. We observe that we may choose s = 0 if $m \equiv 1 \pmod 5$, s = 4 if $m \equiv 0$ or 4 (mod 5) and $s = \frac{4(m-1)}{3}$ if $m \equiv 1 \pmod 3$ (see [4]). We may also apply the following.

Theorem 2.9 [22] There exists a GD[5, 1, {4, 8*}, 4m+8] where $m \equiv 0$ or 2 (mod 5), $m \geq 7$ with the possible exception of m = 10.

Our last recursive construction is the following.

Theorem 2.10 If there exists (1) a RMGD[5, 1, 5, 5m], (2) a GD[5, λ, 4, 4m], (3) a (20+h, 5, λ) covering design with a hole of size h, (4) $\alpha(20+h, 5, \lambda) = \phi(20+h, 5, \lambda)$, then $\alpha(20m+h, 5, \lambda) = \phi(20m+h, 5, \lambda)$.

Proof. Take a RMGD[5, 1; 5, 5m] and inflate this design by a factor of 4, giving a RMGD[5,λ, 20, 20m]. Replace all its groups of size 20 by the blocks of a GD[5, λ, 4, 20]. Add h points to the groups, then on the first m–1 groups construct a (20+h, 5, λ) covering design with a hole of size h and on the last group construct a (20+h, 5, λ) minimal covering design. Finally, on the blocks of size m construct a GD[5, λ, 4, 4m].

We close this section with the following notation that will be used later. A block, <d, d+m, d+n, d+j, f(d)> mod v, where f(d) = a if d is even and f(d) = b if d is odd is denoted by <0 m n j > $\cup$ {a, b} mod v.
Similarly, a block < (0,d) (0,d+m) (1,d+n) (1, d+j) f(d) > mod(–, v) where f(d) = a if d is even and f(d) = b if d is odd is denoted by <(0,0) (0,m) (1,n) (1,j) > $\cup$ {a,b} mod(–, v). When using this notation, a and b are usually infinite points.

3. The Structure of Packing and Covering Designs

Let (V,β) be a (v,k,λ) packing design, for each 2-subset e = {x,y} of V define m(e) to be the number of blocks in b which contain e. Note that by the definition of a packing design we have m(e) $\leq \lambda$ for all e.

The complement of (V,β), denoted by C(V,β) is defined to be the graph with vertex set V and edges e occurring with multiplicity λ -m(e) for all e. The number of edges (counting multiplicities in C(V,β)) is given by $\lambda\binom{v}{2} - |b|\binom{k}{2}$. The degree of a vertex x is $\lambda(v-1) - r_x(k-1)$ where r_x is the number of blocks containing x.

In a similar way we define the excess graph of a (V,β) covering design denoted by E(V,β), to be the graph with vertex set V and edges e occurring with multiplicity m(e)–λ for all e where m(e) $\geq \lambda$ The number of edges in E(V,β) is given by $|b| \binom{k}{2} - \lambda \binom{v}{2}$; and the degree of a vertex v is $r_x(k-1) - \lambda(v-1)$ where r_x is as before.

To define the excess graph of a covering design with a hole H of size h let e = {x,y} where at least one of x or y does not lie in H and let m(e) be the number of blocks in β which contain e. Then the excess graph of the covering design with a hole H of size h, denoted by E(V\H,β), is the graph with vertex set V and edges e occuring with multiplicity m(e) − λ. In a similar way the complement graph, E(V\H,β), of a (v,k,λ) packing design with a hole of size h is defined.

<u>Lemma 3.1</u> [5] Let (V,β) be a (v,5,4) packing design with ψ(v,5,4)−e blocks, where e = 1 if v ≡ 3 (mod 5) and 0 otherwise. Then the degree of each vertex of C(V,β) is divisible by 4 and the number of edges in the graph is 0, 4 or 12 when v mod 5 ∈ {0,1}, {2,4}, or {3}.

The only graph with 4 edges and every vertex of a degree divisible by 4 is the graph with four parallel edges connecting two vertices and v−2 isolated vertices. Therefore, when v ≡ 2 or 4 (mod 5) a (v,5,4) optimal packing design is the same as a (v,5,4) packing design with a hole of size 2.

<u>Lemma 3.2</u> [5] Let (V,β) be a (v,5,2) optimal packing design where v ≡ 3 (mod 10). Then the degree of each vertex of C(V,β) is divisible by 4 and the number of edges in the graph is 6. Hence, C(V,β) consists of v−3 isolated vertices and 3 other vertices the pairs of which are connected by 2 edges.

<u>Lemma 3.3</u> [5] Let (V,β) be a (v,5,4) minimal covering design. Then the degree of each vertex of E(V,β) is divisible by 4 and the number of edges in the graph is 0, 6 or 8 when v mod 5 ∈ {0,1}, {2,4} or {3} respectively.
The only graph with 6 edges and every vertex of a degree divisible by 4 is the graph with v−3 isolated vertices and 3 other vertices each one connected to the other 2 by two parallel edges.

The following is very simple but most useful to us.

<u>Theorem 3.1</u> If there exists

1) A $(v,5,\lambda)$ covering design with $\phi(v,5,\lambda)$ blocks.

2) A $(v,5,\lambda')$ packing design with $\psi(v,5,\lambda')$ blocks.

3) $\phi(v,5,\lambda) + \psi(v,5,\lambda') = \phi(v,5,\lambda+\lambda')$.

4) The complement graph $C(V,\beta)$ of the packing design is isomorphic to a subgraph G of the excess graph $E(V,\beta)$ of the covering design.

Then there exists a $(v,5,\lambda+\lambda')$ covering design with $\phi(v,5,\lambda+\lambda')$ blocks, that is, a $(v,5,\lambda+\lambda')$ minimal covering design.

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<u>Lemma 4.1</u> Let $v \equiv 0$ or 16 (mod 20) be a positive integer. Then $\alpha(v, 5, \lambda) = \phi(v, 5, \lambda)$ for $\lambda > 1$ with the exception of $(v, \lambda) = (56, 5), (56, 9)$.

<u>Proof.</u> If $v \equiv 0$ or 16 (mod 20) then there exists a B[v, 5, 4] [23] . On the other hand for such v, $\alpha(v, 5, \lambda) = \phi(v, 5, \lambda)$ for $\lambda = 2, 3, 4, 5$, by Theorem 1.2 therefore by Lemma 2.1 the result hold except possibly when $v = 56$ and $\lambda \equiv 1 (\text{mod } 4)$. We now construct a (56, 5, 13) minimal covering design and then invoke the previous lemma to get the result.

For a (56, 5, 13) minimal covering design let $X = Z_{48} \cup H_8$ where $H_8 = \{h_1, \ldots h_8\}$. Adjoin a point $\{\infty\}$ to H_8 and on $Z_{48} \cup H_8 \cup \{\infty\}$ take 10 copies of a (57, 5, 1) covering design with a hole of size 9, [22], such that the hole is $H_8 \cup \{\infty\}$. In copy i, i=1, ..., 8, replace "∞" by h_i. In copy 9 replace "∞" by h_1 and in copy 10 replace "∞" by h_2. Furthermore, take the following blocks under the action of the group Z_{48}.

$<0\ 11\ 24\ 35> \cup \{h_1, h_2\}$ half orbit $<0\ 8\ 19\ 29> \cup \{h_7, h_8\}$.

$<0\ 1\ 3\ 7\ 17>$ $<0\ 5\ 15\ 23\ 35>$ $<0\ 1\ 4\ 26\ 32>$ $<0\ 2\ 9\ 31\ 36>$

$<0\ 1\ 3\ 7\ 15>$ $<0\ 5\ 21\ 30> \cup \{h_3, h_4\}$ $<0\ 9\ 20\ 33> \cup \{h_5, h_6\}$

Notice that $\alpha(8, 5, 13) = \phi(8, 5, 13)$ by lemma 5.3.

<u>Lemma 4.2</u> Let $v \equiv 4$ (mod 20) be a positive integer greater than 4. Then there exists a (v, 5, 3) minimal covering design such that there is one pair that appears in at least six blocks. Furthermore, one block in this design can be replaced by a block of size 2 and the covering property still holds.

Proof. The construction of such design is as follows:

1) Take a (v–2, 5, 1) minimal covering design [24, p.50]. This design has a block that can be replaced by a block of size 2, say ⟨v–3, v–2⟩, and the covering property still holds.

2) Take a B[v+1, 5, 1] [23]. Assume in this design we have the block ⟨1 2 3 v v+1⟩ where {1, 2, 3} are arbitrary numbers. In this block change v+1 to v–1, and in all other blocks change v+1 to v.

3) Again take a B[v+1, 5, 1]. Assume in this design we have the block ⟨1 2 3 v–1 v+1⟩. In this block change v+1 to v, and in all other blocks change v+1 to v–1.

It is readily checked that the above three steps yield a (v, 5, 3) minimal covering design such that it has a block of size 2 and the pair {v–1, v} appears at least six times: three times in step 2 and three other times in step 3.

Lemma 4.3 Let $v \equiv 4 \pmod{20}$ be a positive integer greater than 4. Then $\alpha(v, 5, 11) = \phi(v, 5, 11)$.

Proof. For all $v \equiv 4 \pmod{20}$, $v \geq 24$, the construction is as follows:

1) Take a (v, 5, 4) optimal packing design [14]. In this design each pair appears in precisely four blocks except one pair, say, {v–1, v} that appears in zero blocks.

2) Take a (v, 5, 4) minimal covering design [8, 11]. This design has a triple, say, {v–2, v–1, v}, the pairs of which appear in six blocks.

3) Take a (v, 5, 3) minimal covering design as constructed in lemma 4.1. This design has a pair, say, {v–1, v} that appears in at least six blocks.

Now it is readily checked that the above three steps yield a (v, 5, 11) minimal covering design.

Lemma 4.4 $\alpha(v, 5, 11) = \phi(v, 5, 11)$ for v = 8, 28, 48, 68, 88.

Proof. The required constructions are given in the following table. In general, the construction in this table, and all other tables to come, is as follows: Let $X = Z_{v-n} \cup H_n$ or $X = Z_2 \times Z_{\frac{v-n}{2}} \cup H_n$ where $H_n = \{h_1, \ldots, h_n\}$ is the hole. The blocks are constructed by taking the orbits of the tabulated base blocks.

v	Point Set	Base Blocks
8	Z_8	⟨0 1 2 3 4⟩ ⟨0 1 2 4 5⟩ ⟨0 1 2 4 5⟩ ⟨0 1 3 4 6⟩
28	Z_{28}	⟨0 1 2 3 8⟩ twice ⟨0 2 6 15 19⟩ twice ⟨0 3 9 17 21⟩ twice ⟨0 3 10 15 20⟩ twice ⟨0 1 2 5 18⟩ ⟨0 2 9 14 20⟩ ⟨0 3 9 17 21⟩ ⟨0 1 2 3 9⟩ ⟨0 2 5 14 18⟩ ⟨0 3 7 15 21⟩ ⟨0 4 9 15 20⟩
48	$Z_{40} \cup H_8$	Take a (48, 5, 8) covering design with a hole of size 8 [10]. Furthermore, take the following blocks: ⟨0 8 16 24 32⟩ + i, $i \in Z_8$, twice ⟨0 2 4 10 22⟩ ⟨0 1 11 18⟩ $\cup \{h_i\}_{i=1}^{4}$ ⟨0 3 9 26⟩ $\cup \{h_i\}_{i=5}^{8}$ ⟨0 1 5 14 29⟩ ⟨0 1 4 9⟩ $\cup \{h_1, h_2\}$ ⟨0 2 15 21⟩ $\cup \{h_3, h_4\}$ ⟨0 3 10 29⟩ $\cup \{h_5, h_6\}$ ⟨0 5 12 25⟩ $\cup \{h_7, h_8\}$
68	$Z_{60} \cup H_8$	Take two copies of a (68, 5, 4) covering design with a hole of size 8. Such design can be constructed from a T[6, 1, 12] by deleting 6 points from last group, then replace each block of the resultant design by the blocks of a B[v, 5, 4], v = 5, 6. Finally, add 2 points to the groups and on the first 5 groups construct a (14, 5, 4) covering design with a hole of size 2 and take these 2 points with the last group to be the hole of size 8. Furthermore, take the following blocks: ⟨0 12 24 36 48⟩ + i, $i \in Z_{12}$, twice ⟨0 4 10 24 32⟩ ⟨0 1 3 10 40⟩ ⟨0 6 14 32 47⟩ ⟨0 1 2 4 11⟩ ⟨0 7 19 27 44⟩ ⟨0 3 17 38⟩ $\cup \{h_i\}_{i=1}^{4}$ ⟨0 5 23 34⟩ $\cup \{h_i\}_{i=5}^{8}$ ⟨0 4 19 35⟩ $\cup \{h_1, h_2\}$ ⟨0 5 16 43⟩ $\cup \{h_3, h_4\}$ ⟨0 5 18 31⟩ $\cup \{h_5, h_6\}$ ⟨0 6 15 45⟩ $\cup \{h_7, h_8\}$
88	$Z_{80} \cup H_8$	Take two copies of an (88, 5, 4) covering design with a hole of size 8. Such design can be constructed from a T[6, 1, 16] by deleting 8 points from last group, then replacing all its blocks and the first 5 groups by the blocks of a B[v, 5, 4], v = 5, 6, 16, and take the last

group to be the hole. Furthermore, take an (80, 5, 1) minimal covering design [25] and the following blocks: ⟨0 16 32 48 64⟩ + i, $i \in Z_{16}$, twice ⟨0 5 23 38⟩ $\cup \{h_i\}_{i=1}^{4}$

⟨0 7 29 46⟩ $\cup \{h_i\}_{i=5}^{8}$

⟨0 7 15 66⟩ $\cup \{h_1, h_2\}$ ⟨0 9 34 45⟩ $\cup \{h_3, h_4\}$ ⟨0 5 27 52⟩ $\cup\{h_5, h_6\}$ ⟨0 11 35 54⟩ $\cup \{h_7, h_8\}$

⟨0 1 3 9 21⟩ ⟨0 4 14 31 44⟩ ⟨0 1 3 13 41⟩ ⟨0 4 23 54 60⟩

Lemma 4.5 Let $v \equiv 8 \pmod{20}$ be a positive integer. Then $\alpha(v, 5, 11) = \phi(v, 5, 11)$.

Proof. For v = 8, 28, 48, 68, 88, the result follows from lemma 4.3. For $v \geq 108$, $v \neq 128$, simple calculations show that v can be written in the form v = 20m + 4u + h + s where m, u, h and s are chosen so that:

1) there exists a RMGD[5, 1, 5, 5m];
2) there exists a GD[5, 11, {4, s*}, 4m+s];
3) $4u + h + s \equiv 8, 28, 48, 68, 88$;
4) $0 \leq u \leq m-1$, $s \equiv 0 \pmod 4$ and h = 0.

Now apply theorem 2.6 with $\lambda = 11$ and the result follows. For v = 128 apply theorem 2.3 with n = 7, h = 0, and u = 2.

Lemma 4.6 Let $v \equiv 12 \pmod{20}$ be a positive integer. Then $\alpha(v, 5, 11) = \phi(v, 5, 11)$.

Proof. For all positive integers $v \equiv 12 \pmod{20}$, the construction is as follows:

1) Take a (v, 5, 4) optimal packing design [14]. In this design each pair appears in 4 blocks except one pair, say, {a, b} that appears in zero blocks.
2) Take a (v, 5, 4) minimal covering design [11]. In this design there is a triple, say, {a, b, c} the pairs of which appear in 6 blocks.
3) Take a (v, 5, 3) minimal covering design [16]. If this design has a pair, say, {a, b} that appears in 5 blocks then we are done. Otherwise, simple calculation shows that we may assume that {a, b} and {a, 5} appear 4 times in the blocks of the (v, 5, 3) minimal covering design. Assume in design (1) we have the block ⟨1 2 3 a 5⟩ and in design (2) we have the block ⟨1 2 3 b c⟩. In the first block change 5 to b and in the second block change b to 5. Now it is

readily checked that the above construction yields a (v, 5, 11) minimal covering design for $v \equiv 12 \pmod{20}$.

Theorem 4.1 $\alpha(v, 5, 11) = \phi(v, 5, 11)$ for all positive integers $v \equiv 0 \pmod 4$, $v \geq 8$.

5. COVERING WITH INDEX 13

Lemma 5.1 (a) $\alpha(v, 5, 13) = \phi(v, 5, 13)$ for v = 24, 64, 84.

(b) There exists a (24, 5, 13) covering design with a hole of size 4.

Proof. (a) For a (24, 5, 13) minimal covering design the construction is as follows:

1) Take a (24, 5, 7) optimal packing design [10]. The complement graph of this design is a 1-factor, that is a ladder graph on 24 vertices such that the vertices contain all the numbers from 0 to 23.

2) Take the following blocks of a (24, 5, 6) minimal covering design on $X=Z_{24}$

⟨0 1 2 4 10⟩ (mod 24) ⟨0 1 6 12 17⟩ (mod 24)

⟨0 1 2 4 13⟩ (mod 24) ⟨0 1 3 7 19⟩ (mod 24)

⟨0 2 9 12 17⟩ (mod 24) ⟨0 3 8 13 17⟩ (mod 24)

⟨0 3 9 13 17⟩ (mod 24)

The excess graph of the above (24, 5, 6) minimal covering design has a subgraph that is a 1-factor. Now apply theorem 3.1 to get the result.

For v = 64, 84, again take a (v, 5, 7) optimal packing design [10]. The complement graph is a 1-factor. Furthermore, take a (v, 5, 6) minimal covering design as given in [6]. Close observation shows that the excess graph contains a subgraph that is 1-factor. Now apply theorem 3.1 to get the result.

(b) For a (24, 5, 13) covering design with a hole of size 4 proceed as follows:

1) Take two copies of a (23, 5, 2) optimal packing design [9]. In this design each pair appears exactly twice except a triple, say, {21, 22, 23}, the pairs of which appear in zero blocks.

2) Take four copies of a B[25, 5, 1]. Assume that in each copy we have the block ⟨21 22 23 24 25⟩. Delete this block and in all other blocks change 25 to 24.

3) Take a (24, 5, 5) covering design with a hole of size 4 [5]. It is readily checked that the above three steps yield a (24, 5, 13) covering design with a hole of size 4.

Lemma 5.2 Let $v \equiv 4 \pmod{20}$ be a positive integer greater than 4. Then $\alpha(v, 5, 13) = \phi(v, 5, 13)$ with the possible exception of $v = 44$.

Proof. For v = 24, 64, 84, the result is given in lemma 5.1. For $v \geq 124$, $v \neq 144$, 184, 224, 304, simple calculations show that v can be written in the form $v = 20m + 4u + h + s$ where m, u, h, and s are chosen so that:

1) there exists a RMGD[5, 1, 5, 5m];
2) there exists a GD[5, 13, {4, s*}, 4m+s];
3) $4u + h + s = 24, 64, 84$;
4) $0 \leq u \leq m-1$, $s \equiv 0 \pmod 4$ and $h = 4$.

Now apply theorem 2.6 to get the result.

For v = 104, 224, 304, apply theorem 2.10 with m = 5, 11, 15 and $\lambda = 13$. For v = 144 apply theorem 2.8 with m = 7, $\lambda = 13$ and s = h = 0. For v = 184, apply theorem 2.7 with m = 8 and u = 5.

Lemma 5.3. $\alpha(v, 5, 13) = \phi(v, 5, 13)$ for v = 8, 28, 48, 68, 88.

Proof. For v = 8, let $X = Z_5 \cup \{a, b, c\}$. On X construct an (8, 5, 8) optimal packing design [13]. This design has a pair, say, {a, b} that appears in four blocks, and each other pair appears in eight blccks. Furthermore, take the following blocks of an (8, 5, 5) minimal covering design in which the pair {a, b} appears 10 times. To construct this design, let $X = Z_5 \cup \{a, b, c\}$. Then the blocks are ⟨0 2 a b c⟩ (mod 5) ⟨0 1 2 a b⟩ (mod 5) ⟨0 1 2 3 c⟩ (mod 5).
For v = 48, 68, 88 take the blocks of a (v, 5, 8) optimal packing design [13]. This design has a pair, say, {a, b} that appears in 4 blocks while each other pair appears in 8 blocks. Furthermore, take the blocks of a (v, 5, 5) covering design with a hole of size 8 [6] and on the hole of size 8 construct an (8, 5, 5) minimal covering design such that one pair, {a, b}, appears 10 times.
For v = 28 see next table.

v	Point Set	Base Blocks
28	$Z_2 \times Z_{12} \cup H_4$	⟨h1 h2 h3 h4⟩ (orbit length 1)

$\langle(0,0)\ h_1\ h_2\ h_3\ h_4\rangle$ $\langle(1,0)(1,1)(1,3)(1,10)\rangle \cup \{h_i\}_{i=1}^{4}$

$\langle(0,0)(0,1)(0,2)(0,3)(0,5)\rangle$ $\langle(0,0)(0,1)(0,2)(0,6)(0,8)\rangle$

$\langle(1,0)(1,1)(1,2)(1,3)(1,7)\rangle$ $\langle(1,0)(1,1)(1,4)(1,6)(1,9)\rangle$

$\langle(0,0)(0,1)(0,4)(0,7)(1,11)\rangle$ $\langle(0,0)(0,2)(0,5)(0,9)(1,2)\rangle$

$\langle(0,0)(1,1)(1,3)(1,6)(1,8)\rangle$ $\langle(0,0)(0,1)(0,5)(1,0)(1,1)\rangle$

$\langle(0,0)(0,2)(1,0)(1,1)(1,3)\rangle$ $\langle(0,0)(0,1)(0,4)(1,0)(1,6)\rangle$

$\langle(0,0)(0,3)(1,5)(1,7)(1,9)\rangle$ $\langle(0,0)(0,2)(0,5)(1,3)(1,9)\rangle$

$\langle(0,0)(0,4)(1,2)(1,7)(1,10)\rangle$ $\langle(0,0)(0,1)(0,3)(1,5)(1,9)\rangle$

$\langle(0,0)(0,4)(1,3)(1,8)(1,9)\rangle$ $\langle(0,0)(0,1)(1,0)(1,1)\ h_1\rangle$ 4 times

$\langle(0,0)(0,2)(1,3)(1,7)\ h_1\rangle$ twice $\langle(0,0)(0,1)(1,0)(1,1)\ h_2\rangle$

$\langle(0,0)(0,4)(1,2)(1,10)\ h_2\rangle$ $\langle(0,0)(0,5)(1,8)(1,11)\ h_2\rangle$

$\langle(0,0)(0,2)(1,4)(1,8)\ h_2\rangle$ $\langle(0,0)(0,5)(1,7)(1,10)\ h_2\rangle$

$\langle(0,0)(0,6)(1,1)(1,5)\ h_2\rangle$ $\langle(0,0)(0,6)(1,3)(1,9)\ h_3\rangle$

$\langle(0,0)(0,5)(1,2)(1,9)\ h_3\rangle$ 3 times

$\langle(0,0)(0,4)(1,8)(1,10)\ h_3\rangle$ twice

$\langle(0,0)(0,3)(1,6)(1,10)\ h_4\rangle$ $\langle(0,0)(0,5)(1,2)(1,5)\ h_4\rangle$

$\langle(0,0)(0,2)(1,1)(1,7)\ h_4\rangle$ $\langle(0,0)(0,3)(1,6)(1,8)\ h_4\rangle$

$\langle(0,0)(0,4)(1,7)(1,10)\ h_4\rangle$ $\langle(0,0)(0,5)(1,2)(1,10)\ h_4\rangle$

Lemma 5.4 Let $v \equiv 8 \pmod{20}$ be a positive integer. Then $\alpha(v, 5, 13) = \phi(v, 5, 13)$.

Proof. For v = 8, 28, 48, 68, 88, the result follows from lemma 5.3. For $v \geq 108$, $v \neq 128$, write $v = 20m + 4u + h + s$ where m, u, h, and s are chosen as in lemma 4.5. Now apply theorem 2.6 with $\lambda = 13$ to get the result.
For v = 128 apply theorem 2.3 with n = 7, h = 0, and u = 2.

Lemma 5.5 Let $v \equiv 12 \pmod{20}$ be a positive integer. Then $\alpha(v, 5, 13) = \phi(v, 5, 13)$.

Proof. For all positive integers $v \equiv 12 \pmod{20}$ the blocks of a (v, 5, 13) minimal covering design are the blocks of a (v, 5, 9) [7] and a (v, 5, 4) minimal covering design [11].

In this section we have shown

Theorem 5.1 $\alpha(v, 5, 13) = \phi(v, 5, 13)$ for all positive integers $v \equiv 0 \pmod 4$, $v \geq 8$ with the possible exception of $v = 44$.

6. COVERING WITH INDEX 14

Lemma 6.1 Let $v \equiv 4 \pmod{20}$ be a positive integer greater than 4. Then $\alpha(v, 5, 14) = \phi(v, 5, 14)$.

Proof. For all positive integers $v \equiv 4 \pmod{20}$, $v \geq 24$, the construction is as follows:

1) Take a (v, 5, 4) optimal packing design [14]. In this design there is one pair, say, {v–2, v–1} that appears in zero blocks while each other pair appears in four blocks. Furthermore, assume in this design we have the block ⟨9 10 11 v–3 v–1⟩ where 9, 10, 11 are arbitrary numbers. In this block change v–1 to v.

2) Take two copies of a (v, 5, 4) minimal covering design [8, 11]. This design has one triple, the pairs of which appear in six blocks. Assume, in both copies, the triple is {v–3, v–2, v–1}.

3) Take a (v–1, 5, 1) minimal covering design [26]. This design has a block of size 3, say, ⟨v–3, v–2, v–1⟩ which we delete.

4) Take a B[v+1, 5, 1] and assume we have the block ⟨9 10 11 v v+1⟩. In this block change v+1 to v–1 and in all other blocks change v+1 to v.

The above four steps give a design such that the pair {v–3, v–2} appears 17 times, {v–3, v–1} 16 times, and {v–1, v} appears 15 times, the pair {v–2, v–1} appears 13 times, and each other pair appears at least 14 times.

To have the pair {v–2, v–1} appearing 14 times assume in the (v, 5, 4) optimal packing design we have the block ⟨1 2 3 v–2 v–3⟩ where {1, 2, 3} are arbitrary numbers. In this block change v–3 to v–1. And assume in B[v+1, 5, 1] we have the block ⟨1 2 3 v v–1⟩. In this block change v–1 to v–3.

It is easy to check that the above construction yields a (v, 5, 14) minimal covering design for all $v \equiv 4 \pmod{20}$, $v \geq 24$.

Lemma 6.2 $\alpha(v, 5, 14) = \phi(v, 5, 14)$ for v = 8, 28, 48, 68, 88.

Proof: The constructions of these designs are given in the next table.

v	Point Set	Base Blocks
8	Z_8	⟨0 1 2 4 5⟩ twice ⟨0 1 2 3 4⟩ ⟨0 1 2 4 6⟩ ⟨0 1 3 4 6⟩
28	Z_{28}	⟨0 1 2 3 8⟩ 4 times ⟨0 2 6 14 19⟩ 3 times ⟨0 3 10 14 19⟩ 3 times ⟨0 3 10 16 20⟩ 3 times ⟨0 2 6 15 19⟩ ⟨0 3 9 17 21⟩ ⟨0 3 10 15 20⟩ ⟨0 1 2 7 16⟩ ⟨0 2 5 13 17⟩ ⟨0 3 9 13 21⟩
48	$Z_{40} \cup H_8$	On $Z_{40} \cup H_7$ take 5 copies of a (47, 5, 2) covering design with a hole of size 7 [31]. Furthermore, take the following blocks: ⟨0 3 20 23 h_8⟩ half orbit ⟨0 8 16 24 32⟩+i, $i \in Z_8$, 3 times ⟨0 1 10 22 28⟩ ⟨0 2 11 25 h_1⟩ ⟨0 3 7 20 h_2⟩ ⟨0 5 15 25 h_3⟩ ⟨0 1 2 4 h_4⟩ ⟨0 4 9 14 h_5⟩ ⟨0 6 12 25 h_6⟩ ⟨0 7 14 25 h_7⟩ ⟨0 1 3 10 h_8⟩ ⟨0 4 12 25 h_8⟩ ⟨0 5 16 22 h_8⟩.
68	$Z_{60} \cup H_8$	On $Z_{60} \cup H_7$ take 5 copies of a (67, 5, 2) covering design with a hole of size 7 [31]. Furthermore, take the following blocks: ⟨0 5 30 35 h_8⟩ half orbit ⟨0 12 24 36 48⟩+i, $i \in Z_{12}$, 3 times ⟨0 1 3 9 32⟩ ⟨0 4 11 30 44⟩ ⟨0 5 18 33 43⟩ ⟨0 1 3 7 22⟩ ⟨0 8 17 28 42⟩ ⟨0 10 23 39 h_1⟩ ⟨0 1 3 7 h_2⟩ ⟨0 5 14 38 h_3⟩ ⟨0 8 25 40 h_4⟩ ⟨0 10 23 44 h_5⟩ ⟨0 11 23 42 h_6⟩ ⟨0 1 3 4 h_7⟩ ⟨0 4 17 42 h_8⟩ ⟨0 5 15 49 h_8⟩ ⟨0 7 21 40 h_8⟩
88	$Z_{80} \cup H_8$	On $Z_{60} \cup H_7$ take 5 copies of a (87, 5, 2) covering design with a hole of size 7 [31]. Take also the blocks of an (80, 5, 1) minimal covering design on Z_{80} [25]. Furthermore, take the following blocks: ⟨0 13 40 53 h_8⟩ half orbit ⟨0 16 32 44 64⟩+i, $i \in Z_{16}$, 3 times ⟨0 1 3 7 15⟩ ⟨0 5 23 42 51⟩ ⟨0 10 36 49 56⟩ ⟨0 1 3 9 29⟩ ⟨0 10 31 55 h_1⟩ ⟨0 11 33 50 h_2⟩ ⟨0 4 18 39 61⟩ ⟨0 5 36 47 h_3⟩ ⟨0 10 25 63 h_4⟩ ⟨0 1 3 7 h_5⟩ ⟨0 5 13 30 h_6⟩ ⟨0 9 20 43 h_7⟩ ⟨0 12 27 56 h_8⟩ ⟨0 12 30 52 h_8⟩ ⟨0 14 33 59 h_8⟩

Lemma 6.3 Let $v \equiv 8 \pmod{20}$ be a positive integer. Then $\alpha(v, 5, 14) = \phi(v, 5, 14)$.

Proof For v = 8, 28, 48, 68, 88 the result follows from the previous lemma. For v $\geq$ 108 the proof is exactly the same as that of lemma 5.4.

Lemma 6.4 $\alpha(v, 5, 14) = \phi(v, 5, 14)$ for v = 12, 32, 52, 72, 92.

Proof For v = 12 the construction is as follows:

1) Take a (12, 5, 2) minimal covering design as presented in [29]. Take the block ⟨2 6 9 11 12⟩ and change the point 12 to 4. After this change, the pair {9, 12} appears only once, the pairs {2,4}, {9,4} appear four times, the pairs {3,12}, {8,9} appear 3 times and each other pair appears at least twice.

2) Take a (12, 5, 4) minimal covering design [11]. This design has a triple, say, {2, 4, 9} the pairs of which appear in six blocks.

3) Take a (12, 5, 4) optimal packing design [14]. This design has a pair, say, {2, 4} that appears in zero blocks while each other pair appears in four blocks. Furthermore, assume in this design we have the block ⟨1 2 5 12 3⟩ where {1, 5} are arbitrary numbers. In this block change the point 3 to 9.

4) Again, take a (12, 5, 4) optimal packing design, and assume {4, 9} appears in zero blocks. Furthermore, assume we have the block ⟨1 2 5 8 9⟩. In this block change 9 to 3. Now it is easy to check that the above four steps give a (12, 5, 14) minimal covering design.

For v = 32, 52, 72, 92 the construction is as follows:

1) Take a (v, 5, 4) minimal covering design and assume that the pairs of the triple {1, 2, 3} appear in six blocks.

2) Take a (v, 5, 4) optimal packing design and assume that the pair {1, 2} appears in zero blocks.

3) Again take a (v, 5, 4) optimal packing design and assume that the pair {1, 3} appears in zero blocks.

4) Take a (v, 5, 2) covering design with a hole of size 8: For a (32, 5, 2) and (52, 5, 2) covering design with a hole of size 8 see [29], and for a (72, 5, 2) and (92, 5, 2) covering design with a hole of size 8 see [6].

But the (8, 5, 2) minimal covering design has a triple, say, {1, 2, 3} the pairs of which appear in five blocks [29]. It is readily checked that the above four steps yield a (v, 5, 14) minimal covering design for v = 32, 52, 72, 92.

Lemma 6.5 Let $v \equiv 12 \pmod{20}$ be a positive integer. Then $\alpha(v, 5, 14) = \phi(v, 5, 14)$.

Proof For v = 12, 32, 52, 72, 92 the result is given in the previous lemma. For $v \geq 112$, $v \neq 132$, simple calculation shows that v can be written in the form $v = 20m + 4u + h + s$ where m, u, h and s are chosen so that:

1) there exists a RMGD[5, 1, 5, 5m];
2) there exists a GD[5, 14, {4, s*}, 4m+s];
3) $4u + h + s = 12, 32, 52, 72, 92$;
4) $0 \leq u \leq m-1$, $s \equiv 0 \pmod 4$ and $h = 0$.

Now apply theorem 2.6 with $\lambda = 14$ to get the result.
For v = 132 apply theorem 2.3 with n = 7, h = 0, $\lambda = 14$, and u = 3.

In this section we have shown:

Theorem 6.1 Let $v \equiv 0 \pmod 4$ be a positive integer greater than 4. Then $\alpha(v, 5, 14) = \phi(v, 5, 14)$.

7. COVERING WITH INDEX 15

Lemma 7.1 Let $v \equiv 4 \pmod{20}$, $v \geq 24$, be a positive integer. Then $\alpha(v, 5, 15) = \phi(v, 5, 15)$.

Proof For all $v \equiv 4 \pmod{20}$, $v \geq 24$, a (v, 5, 15) minimal covering design can be constructed as follows:

1) Take two copies of a (v, 5, 4) minimal covering design. This design has a triple the pairs of which appear in six blocks. Assume, in both copies, the triple is {a, b, c} [8], [11].
2) Take a (v, 5, 4) optimal packing design. This design has a pair, say, {a, b} that appears in zero blocks while each other pair appears in four blocks.
3) Take a (v, 5, 3) minimal covering design. By lemma 4.1 this design has a block of size two, say, ⟨b, c⟩ which we delete.

Now it is readily checked that the above three steps yield a (v, 5, 15) minimal covernig design for all $v \equiv 4 \pmod{20}$, $v \geq 24$.

Lemma 7.2 Let $v \equiv 8 \pmod{20}$ be a positive integer. Then $\alpha(v, 5, 15) = \phi(v, 5, 15)$.

Proof The blocks of a (v, 5, 15) minimal covering design are the blocks of a (v, 5, 8) and a (v, 5, 7), $v \neq 28$, minimal covering designs.
For $v = 28$ let $X = Z_{26} \cup \{a, b\}$. Then on Z_{26} construct a B[26, 5, 12]. Furthermore, take the following blocks under the action of the group Z_{26}.
⟨0 1 2 3 a⟩ ⟨0 2 5 11 a⟩ ⟨0 3 11 16 a⟩ ⟨0 4 11 16 b⟩
⟨0 4 12 16 b⟩ ⟨0 6 13 19 b⟩ ⟨0 8 17 a b⟩

Lemma 7.3 There exists a (v, 5, 3) covering design with a hole of size 8 for v = 32, 52, 72, 92.

Proof For v = 32, see [16].
For v = 52, 72, 92 see the following table.

v	Point Set	Base Blocks
52	$Z_{44} \cup H_8$	⟨0 2 6 14 24⟩ ⟨0 1 3 7 20⟩ ⟨0 1 2 6 18⟩ ⟨0 9 19 30⟩$\cup\{h_i\}_{i=1}^{4}$ ⟨0 3 10 25⟩$\cup\{h_i\}_{i=5}^{8}$ ⟨0 5 18 31⟩$\cup\{h_1, h_2\}$ ⟨0 8 17 29⟩$\cup\{h_3, h_4\}$ ⟨0 3 11 24⟩$\cup\{h_5, h_6\}$ ⟨0 5 14 21⟩$\cup\{h_7, h_8\}$
72	$Z_{64} \cup H_8$	⟨0 1 3 7 49⟩ 3 times ⟨0 8 19 32 44⟩ 3 times ⟨0 5 26 35⟩$\cup\{h_i\}_{i=1}^{4}$ ⟨0 10 27 41⟩$\cup\{h_i\}_{i=5}^{8}$ ⟨0 5 26 35⟩$\cup\{h_1, h_2\}$ ⟨0 5 26 35⟩$\cup\{h_3, h_4\}$ ⟨0 10 27 41⟩$\cup\{h_5, h_6\}$ ⟨0 10 27 41⟩$\cup\{h_7, h_8\}$
92	$Z_{84} \cup H_8$	⟨0 4 20 35 47⟩ ⟨0 1 3 7 35⟩ ⟨0 5 15 45 63⟩ ⟨0 8 20 44 67⟩ ⟨0 13 27 46 68⟩ ⟨0 1 3 7 15⟩ ⟨0 5 28 38 60⟩ ⟨0 1 3 9 27⟩ ⟨0 10 21 46 64⟩ ⟨0 11 25 42⟩$\cup\{h_i\}_{i=1}^{4}$ ⟨0 11 37 50⟩$\cup\{h_i\}_{i=5}^{8}$ ⟨0 9 30 44⟩$\cup\{h_1, h_2\}$ ⟨0 5 33 50⟩$\cup\{h_3, h_4\}$ ⟨0 7 16 69⟩$\cup\{h_5, h_6\}$ ⟨0 13 32 55⟩$\cup\{h_7, h_8\}$

Lemma 7.4 $\alpha(v, 5, 15) = \phi(v, 5, 15)$ for v = 12, 32, 52, 72, 92.

Proof For v = 12, the construction is as follows:

1) Take two copies of a (12, 5, 4) optimal packing design on $Z_{10} \cup \{a, b\}$. In this design there is one pair that appears in zero blocks while each other pair appears in four blocks. Assume that in both copies this pair is {a, b}.

2) Take a (12, 5, 4) minimal covering design. This design has a triple, say, {a, b, c} the pairs of which appear in six blocks.

3) Take a (12, 5, 3) minimal covering design such that one of its pairs appears in ten blocks. To construct such design let $X=Z_{10} \cup\{a, b\}$ then take the blocks ⟨0 2 4 6 8⟩+i, $i \in Z_2$ ⟨0 1 2 5 9⟩ (mod 10) ⟨0 3 5 a b⟩ (mod 10).

It is easy to check that the above three steps yield the blocks of a (12, 5, 15) minimal covering design .

For v = 32, 52, 72, 92 the construction is as follows:

1) Take two copies of a (v, 5, 4) minimal covering design. This design has a triple the pairs of which appear in six blocks. Assume, in the first design the triple is {0, 2, 4} and in the second the triple is {0, 4, 6}.

2) Take a (v, 5, 4) optimal packing design. In this design there is a pair, say, {0, 4} that appears in zero blocks while each other pair appears in four blocks.

3) Take a (v, 5, 3) covering design with a hole of size 8. On the hole construct an (8, 5, 3) minimal covering design where $X=Z_8$ and blocks are ⟨0 2 4 6⟩+i, $i \in Z_2$, ⟨0 1 3 4 5⟩ (mod 8). Close observation of this design shows that the pairs {0, 4} and {2, 6} appear five times while each other pair appears at least three times. From this design delete the block ⟨0 2 4 6⟩.

Since {0, 4} and {2, 6} appear five times in the blocks of (v, 5, 3) minimal covering design, and since we assume that the pairs of the triples {0, 2, 4} and {0, 4, 6} appear exactly six times, it is easy to see that when we delete the block ⟨0 2 4 6⟩ we actually did not lose any pair and that the above three steps yield a (v, 5, 15) minimal covering design for v = 32, 52, 72, 92.

<u>Lemma 7.5</u> Let $v \equiv 12 \pmod{20}$ be a positive integer. Then $\alpha(v, 5, 15) = \phi(v, 5, 15)$.

<u>Proof</u> For v = 12, 32, 52, 72, 92 the result follows from the previous lemma. For $v \geq 112$ the proof is the same as that of lemma 6.5.

In this section we have shown:

Theorem 7.1 Let $v \equiv 0 \pmod 4$ be a positive integer greater than 4. Then $\alpha(v, 5, 15) = \phi(v, 5, 15)$.

8. COVERING WITH INDEX 17

Lemma 8.1 (a) Let $v \equiv 4 \pmod{20}$ be a positive integer greater than 4. Then $\alpha(v, 5, 17) = \phi(v, 5, 17)$ with the possible exception of $v = 44$.
(b) There exists a (24, 5, 17) covering design with a hole of size 4.

Proof For all positive integers $v \equiv 4 \pmod{20}$, $v \neq 44$, the blocks of a (v, 5, 17) minimal covering design are the blocks of a (v, 5, 4) and a (v, 5, 13) minimal covering design.
(b) For a (24, 5, 17) covering design with a hole of size 4 proceeds as follows:
1) Take 3 copies of a (23, 5, 2) optimal packing design [9]. In this design there is a triple, say, {21, 22, 23} the pairs of which appear in zero blocks.
2) Take 6 copies of a B[25, 5, 1]. Assume in each copy we have the block ⟨21 22 23 24 25⟩ which we delete and in all other blocks change 25 to 24.
3) Take a (24, 5, 5) covering design with a hole of size 4 [5].

Lemma 8.2 $\alpha(v, 5, 17) = \phi(v, 5, 17)$ for v = 8, 48, 68, 88.

Proof The construction of these designs are as follows:
1) Take a (v, 5, 14) minimal covering design (lemma 6.2). Close obsevation of these designs shows that their excess graphs are two 1-factor.
2) Take a (v, 5, 3) optimal packing design [12]. Close observation of these designs shows that their complement graphs are a 1-factor.
Now apply theorem 3.1 to get the result.

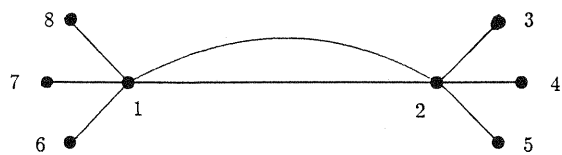
Lemma 8.3 Let $v \equiv 8 \pmod{20}$ be a positive integer. Then $\alpha(v, 5, 17) = \phi(v, 5, 17)$ with the possible exception of $v = 28$.

Proof For v = 8, 48, 68, 88 the result follows from the previous lemma. For $v \geq 108$, $v \neq 128, 168, 208, 268$, write $v = 20m + 4u + h + s$ where m, u, h and s are chosen the same as in lemma 5.2 with the difference that $4u + h + s = 8, 48, 68, 88$. Now apply theorem 2.6 to get the result.
For v = 128 apply theorem 2.3 with n = 7.

For v = 168 apply theorem 2.7 with m = 8 and u = 1.
For v = 208 take a T[5, 17, 40]. Add 8 points to the groups and on the first four groups construct a (48, 5, 17) minimal covering design and on the other groups construct a (48, 5, 17) covering design with a hole of size 8. Such design can be constructed the same as in lemma 8.2 by taking a (48, 5, 14) covering design with a hole size 8 and a (48, 5, 3) packing design with a hole of size 8 [12]. The excess graph of the (48, 5, 14) covering design with a hole of size 8 is a two 1-factor while the complement graph of the (48, 5, 3) packing design is a 1-factor. Now apply theorem 3.1 to get the result.
For v = 268 take a RGD[5, 1, 5, 65] [19] and inflate this design by a factor of 4. To each of 2 parallel classes of blocks size 5 add 4 points and replace their blocks by the blocks of a GD[5, 17, 4, 24]. On the remaining parallel classes construct a GD[5, 17, 4, 20]. Finally, on the groups costruct a (20, 5, 17) minimal covering design. It is clear that this construction yields a (268, 5, 17) covering design with a hole of size 8. hence, $\alpha(268, 5, 17) = \phi(268, 5, 17)$.

Lemma 8.4 $\alpha(v, 5, 17) = \phi(v, 5, 17)$ for v = 12, 32, 52, 72, 92.

Proof For v = 12 the construction is as follows:
1) Take two copies of a (12, 5, 4) optimal packing design [14]. In this design there is one pair that appears in zero blocks while each other pair appears in precisely four blocks. Assume in the first copy the pair is {1, 2} and in the second copy the pair is {2, 3}.
2) Take a (12, 5, 4) minimal covering design. This design has a triple, say, {1, 2, 3} the pairs of which appear in six blocks.
3) Take a (12, 5, 5) minimal covering design [5]. Close observation of this design shows that its excess graph contains the following subgraph.



The above three steps give us a design such that each of its pairs appear in at least 17 blocks except the pair {2, 3} which appears in precisely 16 blocks. To fix this assume in the (12, 5, 4) optimal packing design we have the block ⟨5 6 7 2 4⟩. In this block change 4 to 3. Furthermore, assume in the (12, 5, 4) minimal covering design we have the block ⟨5 6 7 1 3⟩. In this block change 3 to 4. It is readily checked that the above construction yields a (12, 5, 17) minimal covering design.

For v = 32 the construction is as follows:

1) Take a (32, 5, 4) optimal packing design and assume that the pair {1, 3} appears in zero blocks.

2) Take two copies of a (32, 5, 4) minimal covering design. This design has a triple the pairs of which appear in six blocks. Assume in the first copy the triple is {1, 2, 3} and in the second copy the triple is {1, 3, 4}.

3) Take a (32, 5, 5) minimal covering design. This design has a block of size 4, say, ⟨1 2 3 4⟩ [5], which we delete.

4) Assume in the (32, 5, 4) optimal packing design we have the block ⟨5 6 7 3 2⟩ and in the (32, 5, 4) minimal covering design we have the block ⟨5 6 7 4 1⟩. In the first block change 2 to 1 and in the second block change 1 to 2. Now it is easy to check that the above four steps yield the blocks of a (32, 5, 17) minimal covering design.

For v ≥ 52, in [5] we have shown that a (v, 5, 5) minimal covering design with a hole of size 12 or 32 exists. Hence, by invoking the previous constructions, a (v, 5, 17) minimal covering design exists for all v ≡ 12 (mod 20), v ≥ 52.

In this section we have shown:

<u>Theorem 8.1</u> Let v ≡ 0 (mod 4) be a positive integer greater than 8. Then $\alpha(v, 5, 17) = \phi(v, 5, 17)$ with the possible exception of v = 28.

9. COVERING WITH INDEX 18

<u>Lemma 9.1</u> Let v ≡ 4 (mod 20) be a positive integer greater than 4. Then $\alpha(v, 5, 18) = \phi(v, 5, 18)$.

<u>Proof</u> For all v ≡ 4 (mod 20), v ≥ 24, the construction is as follows:

1) Take two copies of a (v, 5, 4) minimal covering design and assume in both copies the pairs of the triple {v–2, v–1, v} appear in six blocks.

2) Take two copies of a (v, 5, 4) optimal packing design. Assume in the first copy the pair {v–2, v–1} appears in zero blocks and in the second copy the pair {v–1, v} appears in zero blocks.

3) Take a (v, 5, 2) minimal covering design. It is readily checked that the above three steps yield a (v, 5, 18) minimal covering design for all $v \equiv 4 \pmod{20}$, $v \geq 24$.

Lemma 9.2 Let $v \equiv 8 \pmod{20}$ be a positive integer. Then $\alpha(v, 5, 18) = \phi(v, 5, 18)$.

Proof A (v, 5, 18) minimal covering design, $v \equiv 8 \pmod{20}$ can be constructed as follows:

1) Take a (v, 5, 8) minimal covering design. This design has a triple, say, {a, b, c} the pairs of which appear in ten blocks [8].

2) Take a (v, 5, 8) optimal packing design. This design has a pair, say, {a, b} that appears in four blocks while each other pair appears in eight blocks [13].

3) Take a (v, 5, 2) minimal covering design. Simple calculation shows that the number of repeated pairs in this design is greater than v. If this design has a pair, say, {a, b} that appears at least four times, then the above three steps give a (v, 5, 18) minimal covering design and we are done. Otherwise, we may assume that the pairs {a, b} and {a, 4} appear three times where 4 is an arbitrary number. In this case the above three steps give a design where each pair appears at least 18 times except the pair {a, b} which appears only 17 times. To have {a, b} appear at least 18 times assume in the (v, 5, 2) minimal covering design we have the block ⟨1 2 3 4 a⟩ where {1, 2, 3} are arbitrary numbers. In this block change a to c.

Furthermore, assume in the (v, 5, 8) optimal packing design we have the block ⟨1 2 3 b c⟩. In this block change c to a. Now it is easy to check that the above construction yields a (v, 5, 18) minimal covering design.

Lemma 9.3 $\alpha(v, 5, 18) = \phi(v, 5, 18)$ for v = 12, 32, 52, 72, 92.

Proof The construction of these minimal covering designs is as follows:

1) Take a (v, 5, 11) optimal packing design [10]. Close observation of these designs shows that their complement graphs are 1-factor.

2) Take a (v, 5, 7) minimal covering design. Close observation of these designs shows that their excess graphs contain a subgraph that is 1-factor. But $\phi(v, 5, 7) + \psi(v, 5, 11) = \phi(v, 5, 18)$, hence, by theorem 3.1, $\alpha(v, 5, 18) = \phi(v, 5, 18)$.

Lemma 9.4 Let $v \equiv 12 \pmod{20}$ be a positive integer. Then $\alpha(v, 5, 18) = \phi(v, 5, 18)$.

Proof For v = 12, 32, 52, 72, 92 the result follows from lemma 9.3. For $v \geq 112$ the proof is the same as that of lemma 6.5.

In this section we have shown:

Theorem 9.1 Let $v \equiv 0 \pmod{4}$ be a positive integer greater than 4. Then $\alpha(v, 5, 18) = \phi(v, 5, 18)$.

10. COVERING WITH INDEX 19

Lemma 10.1 Let $v \equiv 0 \pmod{4}$ be a positive integer greater than 4. Then $\alpha(v, 5, 19) = \phi(v, 5, 19)$.

Proof The blocks of a (v, 5, 19) minimal covering design, $v \equiv 4$, 8 or 12 (mod 20), $v \neq 44$, are the blocks of a (v, 5, 6) and a (v, 5, 13) minimal covering design. Since a (44, 5, 13) minimal covering design is still unknown, we need to construct a (44, 5, 19) minimal covering design. For this purpose, let $X = Z_{44}$, then take the following base blocks under the action of the group Z_{44}.

⟨0 1 2 4 8⟩ 5 times, ⟨0 3 12 19 32⟩ 5 times, ⟨0 5 14 26 31⟩ 5 times
⟨0 6 14 23 33⟩ 5 times, ⟨0 1 3 18 25⟩ 4 times, ⟨0 4 14 23 28⟩ 4 times
⟨0 6 13 24 36⟩ 4 times, ⟨0 1 4 10 32⟩ ⟨0 1 3 9 13⟩ ⟨0 2 16 24 29⟩
⟨0 5 11 22 34⟩ ⟨0 1 2 5 17⟩ ⟨0 1 3 5 11⟩ ⟨0 3 9 25 33⟩ ⟨0 4 15 22 31⟩
⟨0 5 13 22 34⟩.

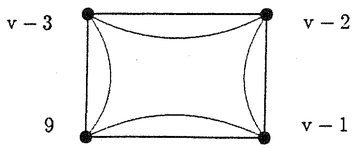
In this section, we have shown:

Theorem 10.1 Let $v \equiv 0 \pmod{4}$ be a positive integer greater than four. Then $\alpha(v, 5, 19) = \phi(v, 5, 19)$.

11. COVERING WITH INDEX 21

Since covering design with index one and $v \equiv 0 \pmod 4$, $v \geq 8$, is far from being settled, it is worth looking at covering designs with index 21 and $v \equiv 0 \pmod 4$.

Lemma 11.1 There exists a (v, 5, 12) minimal covering design for all $v \equiv 4 \pmod{20}$ such that the excess graph consists of v–4 isolated vertices and the following graph on the remaining four vertices.



Proof For all $v \equiv 4 \pmod{20}$ $v \geq 24$ the construction is as follows.

1) Take a (v,5,4) optimal packing design [14] and assume that the pair {9, v-2} appears in zero blocks.

2) Take two copies of a (v,5,4) minimal covering design [8 ,11]. This design has a triple the pairs of which appear in six blocks. Assume that in the first copy the triple is {9, v–1, v–2} and in the second copy the triple is {9, v–2, v–3}.

Lemma 11.2 (a) There exists a (24, 5, 21) covering design with a hole of size 4.
(b) $\alpha(v, 5, 21) = \phi(v, 5, 21)$ for v = 24, 44, 64, 84.

Proof

(a) For a (24, 5, 21) covering design with a hole of size 4 proceed as follows:

1) Take a (24, 5, 5) covering design with a hole of size 4, [5].

2) Take four copies of a (23, 5, 2) packing design with a hole of size 3, [8].

3) Take eight copies of a B[25, 5, 1] and in each copy assume we have the block ⟨21 22 23 24 25⟩ which we delete and in all other blocks we change 25 to 24.

(b) For v = 24, let $X = Z_{20} \cup H_4$. Then the blocks are:

1) $\langle h_1\ h_2\ h_3\ h_4 \rangle$

2) Adjoin a point "∞" to X and on $X \cup \{\infty\}$ costruct 12 copies of a B[25, 5, 1] such that $\langle h_1\ h_2\ h_3\ h_4\ \infty \rangle$ is a block, which we delete. In the first 3 copies of

B[25, 5, 1] replace "∞" by h_1, in the second 3 copies replace "∞" by h_2, in the third 3 copies replace "∞" by h_3 and in the last 3 copies replace "∞" by h_4.

3) Furthermore, take the following base blocks under the action of the group Z_{20}.

⟨0 4 8 12 16⟩+i, i∈Z_4, three times.

⟨0 h_1 h_2 h_3 h_4⟩ ⟨0 1 2 3 7⟩ ⟨0 1 3 10 13⟩ ⟨0 2 6 10 15⟩ ⟨0 1 2 7 10⟩

⟨0 2 7 11 14⟩ ⟨0 1 2 4 h_1⟩ ⟨0 1 4 10 h_2⟩ ⟨0 2 7 15 h_3⟩ ⟨0 3 8 14 h_4⟩

⟨0 3 9 14⟩∪$\{h_i\}_{i=1}^{4}$.

For v = 44, 64, 84 the construction is as follows:

1) Take a (v–1, 5, 1) minimal covering design with a hole of size 3, say, {v–3, v–2, v–1}, [26]. The excess graph of these designs contain a subgraph which is 1–factor on v–4 points. In addition to the 1-factor,assume that {4,5} appears one more time. Furthermore, assume in this design we have the block ⟨1 2 3 9 v–1⟩ where {1, 2, 3, 9} are arbitrary numbers. In this block change v–1 to v.

2) Take a B[v+1, 5, 1] and assume in this design we have the block ⟨1 2 3 v v+1⟩. In this block change v+1 to v–1 and in all other blocks change v+1 to v.

3) Take a (v–2, 5, 1) optimal packing design, [12]. The complement graph of this design is a 1-factor. We may assume that the 1-factor contains (v–3, v–2) and another $\frac{(v-4)}{2}$ pairs on the remaining v–4 points. Furthermore, we may assume that these $\frac{(v-4)}{2}$ pairs of the 1–factor are precisely the 1-factor in the excess graph of the design in (1).

4) Take a B[v+1, 5, 1] and assume we have the block ⟨1 2 3 v v+1⟩ where {1, 2, 3} are arbitrary numbers. In this block change v+1 to v–1 and in all other blocks change v+1 to v.

5) Again take a B[v+1, 5, 1] and assume we have the block ⟨1 2 3 v–1 v+1⟩. In this block change v+1 to v and in all other blocks change v+1 to v–1.

6) Take a (v, 5, 4) optimal packing design [14]. In this design each pair appears exactly 4 times except one pair, say, {v–1, v} which appears in zero blocks.

7) Take a (v, 5, 12) minimal covering design such that its excess graph is the same as in lemma 11.1.

The above seven steps gives us a design such that the pair {v–3, v–1} appears twenty times {4,5} {9,v–3}, {9,v–1}, {v–2, v–1} at least twenty two times and each other pair appears at least twenty one times.

To have (v–3, v–1) appear twenty one times assume in designs (1) and (2) we have the blocks <a b c 5 v–3> <a b c 9 v–1> where {a b c} are arbitrary numbers. In the first block change v–3 to 9 and in the second block change 9 to v–3. But in this case the pair {5, v–3} appears only twenty times. To fix this, assume in design (4) and (5) we have the blocks <d e f 5 4> and <d e f v-3 9> where {d e f} are arbitrary numbers. In the first block change 4 to v–3 and in the second block change v–3 to 4.

For all other values of v, the proof is the same as lemma 5.2.

Lemma 11.3 Let $v \equiv 8$ or $12 \pmod{20}$ be a positive integer. Then $\alpha(v, 5, 21) = \phi(v, 5, 21)$.

Proof For $v \equiv 8 \pmod{20}$ the blocks of a (v, 5, 21) minimal covering design are the blocks of a (v, 5, 13) and (v, 5, 8) minimal covering design.
For $v \equiv 12 \pmod{20}$ the blocks of a (v, 5, 21) minimal covering design are the blocks of a (v, 5, 16) and (v, 5, 5) minimal covering design.

In this section we have shown:

Theorem 11.1 Let $v \equiv 0 \pmod{4}$ be a positive integer. Then $\alpha(v, 5, 21) = \phi(v, 5, 21)$.

12. CONCLUSION

To conclude our result, we have shown (theorem 4.1 - theorem 11.1) that $\alpha(v, 5, \lambda) = \phi(v, 5, \lambda)$ for all $v \equiv 4 \pmod{20}$, $v \equiv 0 \pmod{4}$, $v > 4$, provided $11 \leq \lambda \leq 21$ with the possible exceptions of (v, λ) = (44, 13), (28, 17), (44, 17).

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