BEACHES, BAYS AND HEADLANDS

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Abstract

We consider local and global combinatorial constraints on embeddings of a simple closed curve in a rectangular array of unit squares in which the curve lies entirely within the rectangular array and passes through each unit square once. Geographical terminology is convenient to describe the embeddings. We study shortest segments of the embedded curve with no clockwise quarter turns (bays) and a prescribed number of anticlockwise quarter turns (headlands). We also study embeddings which either maximise or minimise the number of quarter turns. Enumerative questions related to these embeddings have recently been studied by Kwong and Rogers, as Hamiltonian cycles in grid graphs.

1 Introduction

Consider the problem of embedding a simple closed curve in an $m \times n$ rectangular array of unit squares so that the curve lies entirely within the rectangular array and passes through each unit square once (Figure 1). For simplicity we shall require that the curve does not pass through the corner of any of the unit squares. What characteristics must such an embedding have?

The analogy with geographical maps is strong, so geographical terminology conveniently comes to our aid. The interior of the simple closed curve is *land*, the exterior is *sea*. The simple closed curve itself is *coastline*, outlining a *continent*. The unit squares are *quadrates*. The rectangular array of quadrates, with the continent in place, is a *chart*. To measure distances on our chart let us call the length of one side of a quadrate a *league* and the area of one quadrate a *quad*, so $1\,quad = 1\,square\,league$.

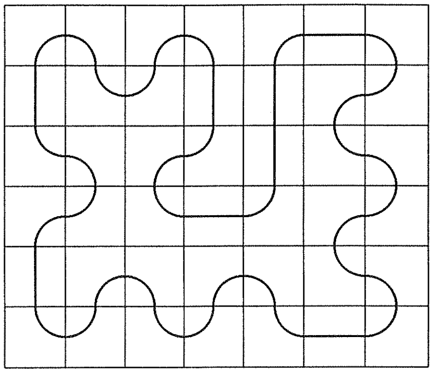

Figure 1: A simple closed curve embedded in a 6×7 array of unit squares.

The coastline within a quadrate can be one of only three essentially different types: when described in the positive sense, the coastline enters the quadrate through one side (the *entrance*) and emerges through one of the other three sides (the *exit*), since it cannot re-enter the quadrate it has just left (Figure 1). If the exit is on the left of the entrance, the coastline in the quadrate is a *headland* (H). If the exit is on the right of the entrance, the corresponding coastline is a *bay* (B), and if the exit is opposite the entrance, the corresponding coastline is a *beach* (S): we choose the symbol S because it is a Straight Sandy Stretch of Shoreline. For simplicity we restrict each of these portions of coastline to be a standard smooth curve, chosen so that each quadrate contains one league of coastline, and a quarter quad of land when the coastline is a headland, a half quad of land when the quad is a beach, and three-quarters of a quad of land when the coastline is a bay.

We orient our charts so that an $m \times n$ chart has m leagues of *latitude* in the north-south direction) and n leagues of *longitude* (in the east-west direction). So that the coastline can indeed form a simple closed curve, we require $m \geq 2$ and $n \geq 2$.

Three consecutive headlands can only occur if the next quadrate also contains a headland. The result is a continent of area 1 quad and coastline of length 4 leagues on a 2×2 chart. In every larger chart, at most two headlands can occur consecutively. Similarly if three consecutive bays occurred on a coastline, the next quadrate would also have to contain a bay; this would lead to a simple closed curve oriented clockwise, surrounded by land and having a lake in its interior. We reject this possibility as being inconsistent with the chosen orientation of the simple closed curve, and conclude that no chart has three consecutive bays.

For an $m \times n$ continent, the coastline can be represented as a (cyclic) word of mn symbols from the alphabet $\{B,H,S\}$, subject to certain restrictions which we have already started to notice, such as not containing the block B^3, and not containing the block H^3 if $mn > 4$.

We adopt the convention that the coastline of a continent is described in the positive sense, usually beginning at the quadrate in the north-west corner of the chart. For example, the continent in Figure 1 has coastline

$$HSHB^2HSH^2B^2H^2B^2HSH^2B^2H^2B^2H^2SHS^2BSB^2HSH^2B^2H.$$

Let b, h, and s denote the number of bays, headlands and beaches in any coastline. The continent in Figure 1 has $b = 15$, $h = 19$, $s = 8$.

2 Some fundamental properties of coastlines

How are the parameters m, n, b, h and s related? First note that the total number of quadrates in an $m \times n$ chart is mn, and each contains just one bay, headland or beach, so

$$b + h + s = mn. \quad (1)$$

We now show that this sum is even.

Theorem 1 *For any $m \times n$ chart, mn is even.*

Proof. Colour the quadrates of the chart black and white, in checkerboard fashion. The coastline passes alternately through black and white quadrates. Since it passes through every quadrate the total number mn must be even. □

Theorem 2 *For any chart, $h - b = 4$.*

Proof. The winding number of a simple closed curve about an interior point is 1. In each headland the coast makes a quarter turn anticlockwise, and in each bay a quarter turn clockwise, so the number of headlands must exceed the number of bays by 4. □

Theorem 2 implies $h \geq 4$. Indeeed the four corner quadrates of a chart can only contain headlands. If these were the only headlands the coastline could not enter any interior quadrates, so $h \geq 5$ must hold whenever $m \geq 3$ and $n \geq 4$. In fact $h \geq 6$.

Theorem 3 *For any chart, s is even.*

Proof. From equation (1) and Theorem 2, $s = mn - 2h + 4$. But mn is even, by Theorem 1, so s is even. □

Thus, a continent has an even number of beaches. Also, the number of headlands has the same parity as the number of bays, by Theorem 2, but these numbers can be odd (Figure 1) or even (Figure 2).

Theorem 4 *The area of the continent in an $m \times n$ chart is $\frac{1}{2}mn - 1$, and the length of the coastline is mn leagues.*

Proof. Summing the land area over all quadrates, the area A of the continent is

$$A = \frac{h}{4} + \frac{3b}{4} + \frac{s}{2}.$$

By (1), this is

$$A = \frac{mn}{2} + \frac{b}{4} - \frac{h}{4}.$$

Now Theorem 2 gives the stated result for the area. The length of coastline is immediate from the requirement that the length in each quadrate is 1 league. □

Theorem 5 *The number of lattice points on the continent of an $m \times n$ chart is $\frac{1}{2}mn - 1$.*

Proof. Each lattice point on land is incident with 4 adjacent quadrates. A quadrate which is incident with x lattice points on land contains a headland, a beach or a bay when $x = 1, 2$ or 3 respectively, and in each case contains land of area $x/4$ quads. It follows that the total land area equals the total number of lattice points on land, and the claimed result follows from Theorem 4. □

Corollary 1 *An $m \times n$ chart has $\frac{1}{2}(m+2)(n+2)$ lattice points in the ocean, $2m+2n$ of which are on the edge of the chart.*

3 Local constraints

Coastlines are subject to both local and global constraints. The global constraints correspond to restrictions on the values of b, h and s, while local constraints correspond to restrictions on the strings of symbols from $\{B,H,S\}$ which are possible blocks in the cyclic word for some chart. The local constraints reflect the requirements that the coastline must pass through every quadrate, without self-intersection.

An *admissible* block is a string of symbols from $\{B,H,S\}$ which occurs in a cyclic word for some chart. An admissible block is *general* if it occurs in the cyclic words of arbitrarily large charts; otherwise it is *special.* For example, the admissible block H^2 is general, whereas H^3 is special, since it only occurs in the cyclic word H^4. Again, the admissible block $HSHS^2HSH$ is special, since it only occurs in the cyclic word $HSHS^2HSH^2B^2H$.

As noted in the previous section, the cyclic word for any chart contains at least four H's. This minimum is achieved when $m = 2$: the unique $2 \times n$ chart has cyclic word $H^2S^{n-2}H^2S^{n-2}$. Any chart with $m \geq 3$ requires at least five H's and one B.

We shall consider just one form of local constraint: which are the shortest general B-free blocks which contain exactly r occurrences of H? (We do not study special blocks here since they correspond to exceptions, whereas general blocks correspond to typical local constraints.) The blocks H and H^2 are the unique solutions for $r = 1$

and $r = 2$ respectively. We have already seen that H^3 is not a general block; the shortest general B-free blocks with $r = 3$ are H^2SH and HSH^2. These two blocks are reversals of each other. We regard them as equivalent since a rotation, followed by a reflection, transforms the coastline corresponding to one into the coastline corresponding to the other. It is straightforward to verify that $H^2S^2H^2$ is the unique shortest general B-free block when $r = 4$.

Lemma 1 *Let $HS^{a_1}HS^{b_1}HS^{a_2}HS^{b_2}H$ be an admissible block. If $a_1 \geq a_2 - 1$ then $b_1 \geq b_2 + 2$.*

Proof. Suppose $a_1 \geq a_2 - 1$. The corresponding spiral portion of coastline has consecutive straight stretches of lengths a_1, b_1, a_2 and b_2. The alternate pairs are parallel and bound an $a \times b$ rectangle of quadrates, where $a = \min\{a_1, a_2\}$ and $b = b_1$. The continuation of the coastline passes through all quadrates in this rectangle, and there must be at least two quadrates on the boundary of the rectangle where it connects with coastline outside the rectangle. Since no quadrates adjacent to the straight stretch of coastline of length b_2 are available for this, $b \geq b_2 + 2$. □

Lemma 2 *Let $HS^{a_1}HS^{b_1}HS^{a_2}HS^{b_2}HS^{a_3}HS^{b_3}H$ be an admissible block with $a_2 \geq a_3$ and $b_1 \leq b_2$. Then at least one of the following holds:*
(1) $a_2 \geq a_1 + a_3 + 4$; (2) $b_2 \geq b_1 + b_3 + 4$.

Proof. Since $a_2 \geq a_3$ we have $b_2 \geq b_3 + 2$ by Lemma 1. Similarly $b_1 \leq b_2$ implies $a_1 + 2 \leq a_2$ by the contrapositive of Lemma 1. Thus the corresponding portion of coastline is a double spiral with six straight stretches, of maximum length a_2 in the a-direction and b_2 in the b-direction. Relative to the first headland, the position vector of the seventh headland is $(a_1 - a_2 + a_3 + 1, b_1 - b_2 + b_3 + 1)$, where the first component is in the a-direction (positive in the sense of the first straight stretch) and the second component is in the b-direction (positive in the sense of the second straight stretch). The continuation of the coastline passes through all quadrates within the double spiral, so there must be at least two quadrates in a line between the first and seventh headlands for the coastline to connect with the region outside the double spiral. Such a pair of quadrates is present in the a-direction if $a_1 - a_2 + a_3 + 1 < -2$, whence (1). If this does not hold, there must be an appropriate pair of quadrates in the b-direction, which requires $b_1 - b_2 + b_3 + 1 < -2$, whence (2). □

Together Lemmas 1 and 2 enable us to solve our local constraint problem. They show that the sequence of lengths of alternate straight stretches of an admissible B-free block must be monotonic or unimodal, and if both are unimodal their maxima are adjacent. The shortest admissible block with a fixed number of H's has both the sequences unimodal, with maxima as nearly central as possible. We omit the details of the proof.

Theorem 6 *A shortest general B-free block with exactly r occurrences of H, for $r \geq 5$, is*

$$H^2SHS^2HS^3...HS^kHXHS^kH...S^3HS^2HSH^2,$$

where $X = S^{k+1}HS^{2k+3}$ *if* $r = 2k+5$, $k \geq 0$, *and* $X = S^{2k+2}$ *if* $r = 2k+4$, $k \geq 1$.

When r *is odd, the block is unique up to reversal, and has length* $(k+3)^2$*; when* r *is even, the block is unique, and has length* $(k+2)(k+3)$.

4 Charts with many beaches

In the rest of this paper we will study the range of values that b, h and s can take for charts of a given dimension. Notice that if we know the value of any one of b, h and s, the values of the other two easily follow from equation (1) and Theorem 2. Thus it's sufficient to consider the range of values that s can take. It will be convenient in what follows to use the word *turn* to mean either a bay or a headland. When the number of turns, $b+h$ is minimized, the number of beaches s is maximised. In this section we study this case. All maps will be $m \times n$, and we will assume without loss of generality that $m \leq n$.

For $k \leq m$, a $k \times k$ *corner square* of a chart is a $k \times k$ square block of quadrates one (or more) of which is a corner quadrate of the chart. Such a quadrate is a *chart corner* of the $k \times k$ corner square (and is unique when $k < m$).

Theorem 7 *A* $k \times k$ *corner square contains at least* k *turns.*

Proof. We use induction on k. The coastline in any corner quadrate of the chart must be a headland, so the theorem holds when $k = 1$. Now suppose it holds for some $k-1$, with $k > 1$. Consider a $k \times k$ corner square and let C be be the coastline in the corner quadrate of the square which is diagonally opposite its chart corner. Then C is either a turn or a beach. In the case $C = S$, the set of k boundary quadrates of the corner square in the line of this beach must contain a turn since the coastline cannot leave the chart. Thus in every case the $k \times k$ corner square contains at least one more turn than the $(k-1) \times (k-1)$ corner square nested within it, which contains at least $k-1$ turns by the induction hypothesis. □

Theorem 8 *An* $m \times n$ *chart with* $m \leq n$ *contains at least* $2m$ *turns. Whenever* m *is even and* $2 \leq m \leq n$*, there is an* $m \times n$ *chart with exactly* $2m$ *turns*

Proof. Case (i): $m = 2k$. Each of the four $k \times k$ corner squares in the chart contains k turns, by Theorem 7, so the chart contains at least $4k = 2m$ turns.

Case (ii): $m = 2k+1$. Now n must be even, by Theorem 1, and so strictly greater than m. Therefore the $k \times k$ corner squares in the northwest and southeast corners and the $(k+1) \times (k+1)$ corner squares in the northeast and southwest corners do not intersect. By Theorem 7 these corner squares contain at least $4k+2 = 2m$ turns.

Thus $2m$ is a lower bound in each case. When $m = 2k$, the bound is achieved by a continent with k peninsulas as shown in Figure 3 for the case $m = 6$. In this construction it is clear that n can be changed without altering the number of turns, so $2m$ turns are achieved whenever m is even and $2 \leq m \leq n$. □

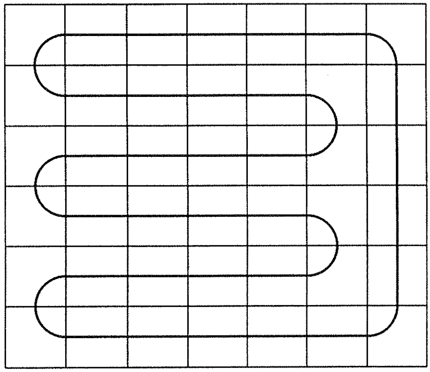

Figure 2: A 6×7 chart with 12 turns

We say a maximal row or column of quadrates is a *line* of the chart. The line is *odd* or *even* according to the parity of the number of quadrates it contains.

Theorem 9 *Any line contains an even number of turns. If the line is even, then it contains an even number of beaches. If the line is odd, it contains at least two turns.*

Proof. Each connected component of coastline contained in the line is either a beach traversing the line or begins and ends with a turn, so the total number of turns is even. If the line is even it follows that it contains an even number of beaches. Now consider an odd line. If the line contained no turns each component of land lying within it would be bounded by a pair of beaches traversing the line. Then the number of beaches would be even, and every quadrate would contain a traversing beach. But the line is odd, a contradiction. Hence the line contains at least one turn. Since the line contains an even number of turns, there are at least two turns. □

Theorem 10 *An $m \times n$ chart with $m < n$ and m odd contains at least $2n$ turns. Whenever $m < n$ and m odd there is an $m \times n$ chart with exactly $2n$ turns.*

Proof. The chart may be partitioned into n lines, each containing m quadrates. Since m is odd, Theorem 9 implies each north-south line contains at least 2 turns, giving $2n$ turns altogether. If $n = 2k$ a construction similar to that in Figure 2, with k peninsulas pointing north rather than west, attains the bound. □

5 Charts with few beaches

In this section we obtain lower bounds on s in an $m \times n$ chart. As in the last section we assume $m \leq n$. We find that the minimum depends on whether m is odd,

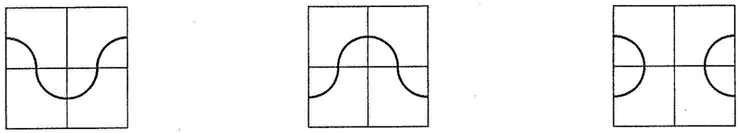

Figure 3: 2×2 beach-free squares in which the coastline does not intersect the horizontal boundaries.

congruent to 0 modulo 4 or to 2 modulo 4, and also to which of these 3 classes n belongs. We cannot have both m and n odd so there are $3^2 - 1 = 8$ combinations to consider. We obtain lower bounds on s in each of these 8 cases, and show that in 4 of them the bounds are best possible. The other cases are open: our bounds are not best possible.

We say a set of k adjacent lines running north-south is an $m \times k$ *zone* and a set of k lines running west to east is a $k \times n$ zone.

Theorem 11 *If $m \equiv 2 \pmod 4$, any $m \times 3$ zone contains at least 2 beaches, and similarly if $n \equiv 2 \pmod 4$, any $3 \times n$ zone contains at least 2 beaches*

Proof. Suppose some $m \times 3$ zone contain no beaches. Partition its two western lines into 2×2 squares, totalling $\frac{1}{2}m$ in all. Consider the northernmost 2×2 square. The coastline cannot cross its northern edge as this is a boundary of the chart. The only 2×2 squares which contain no beaches and in which the coastline does not cross the northern boundary are shown in Figure 3. Note that in each of these the coastline does not cross the southern boundary either. It follows that the coastline does not cut the northern boundary of the next 2×2 square to the south, so this also is one of those in Figure 3. Continuing in this way we see that each 2×2 square in the partition is a copy of one of those in Figure3. In each of the first two of these the coastline crosses the north-south bisector of the squares. It must do this an even number of times, for each time the coastline crosses the bisector it must return. Since $\frac{1}{2}m$ is odd, at least one of the 2×2 squares is of the third type in Figure 3. The western half of this square is not the eastern half of any square in Figure 3 so the eastern $m \times 2$ zone of the chosen $m \times 3$ zone contains a 2×2 square not appearing in Figure 3. This is impossible if the $m \times 3$ zone contains no beaches. Hence the $m \times 3$ zone contains at least one beach. By Theorem 9 the number of beaches in an $m \times 3$ zone is even and so at least 2. □

Theorem 12 *Any $2k \times 2k$ corner square contains at least k beaches.*

Proof. We use induction on k. It is easily seen that the theorem holds when $k = 1$. Now suppose it holds for some $k - 1$, with $k > 1$. Consider the L-shaped set of quadrates which are in a $2k \times 2k$ corner square but not in the $2(k-1) \times 2(k-1)$ corner square within it. Partition this set into $2k - 1$ 2×2 squares, and suppose that none contains a beach. As in the proof of Theorem 11 each 2×2 square in the

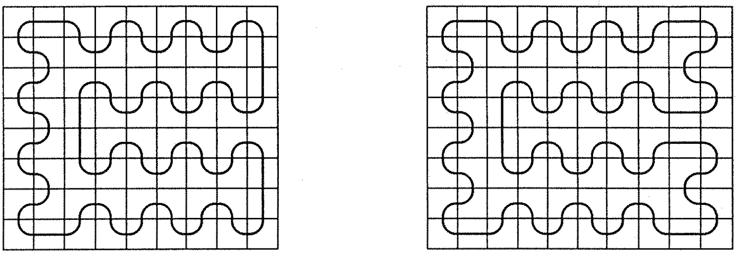

Figure 4: Charts with $m \equiv 0$ $(mod\ 4)$ which attain the bound of Theorem 13.

northsouth arm of the L-shaped region must be one of those in Figure 3. A similar argument shows that each 2×2 square in the eastwest of the L-shaped region is a copy of one of those in Figure 3, but rotated through 90 degrees. Thus the 2×2 square in the corner of the region is one of those in Figure 3, and also a copy of one of these rotated through 90 degrees. This is impossible so we conclude the L-shaped region contains at least beach. Thus in every case the $2k \times 2k$ corner square contains at least one more beach than the $2(k-1) \times 2(k-1)$ corner square nested within it, which contains at least $k-1$ beaches by the induction hypothesis. □

Theorem 13 *An $m \times n$ chart with $m \equiv 0$ $(mod\ 4)$ has at least m beaches. Whenever $m \equiv 0$ $(mod\ 4)$ there is an $m \times n$ chart with exactly m beaches.*

Proof. Let $m = 4k$. Each $2k \times 2k$ corner square of the chart contains k beaches, by Theorem 12, so the chart contains at least $4k = m$ beaches.

To see that this bound is attained consider the two charts in Figure 4, one 8×9, the other 8×10. Each contains 8 beaches and extending each shows that this number can be held fixed while n is increased in multiples of 2. Clearly, m can also be increased or decreased in multiples of 4 while keeping the number of beaches equal to m. □

Theorem 14 *An $m \times n$ chart with m odd has at least n beaches. Whenever $m \equiv 1$ (mod 2), $n \equiv 0$ $(mod\ 4)$ there is an $m \times n$ chart with exactly n beaches.*

Proof. By Theorem 9, each north-south line contains at least one beach so we have at least n beaches altogether. If $n \equiv 0$ $(mod\ 4)$ a construction like that of Figure 4 achieves this bound. □

Theorem 15 *An $m \times n$ chart with $m \equiv 2$ $(mod\ 4)$ contains at least $m+2+2\lfloor(n-m-2)/3\rfloor$ beaches whenever $n \geq m+2$, and at least $m+2$ beaches whenever $n = m$ or $n = m+1$.*

Proof. Let $m = 4k+2$. First suppose that $n \geq m+2$. The $2k \times 2k$ corner square in the northwest corner contains at least k beaches and the $(2k+2) \times (2k+2)$ corner square in the southwest contains at least $k+1$ beaches. This gives a total of at least $2k+1$ beaches in the westernmost $m \times (2k+2)$ zone, but the number of beaches there must be even by Theorem 9 so in fact we have at least $2k+2$ beaches. Similarly we have at least $2k+2$ beaches in the easternmost $m \times (2k+2)$ zone, giving at least $4k+4 = m+2$ beaches so far. Between the regions so far considered there are at least $\lfloor (n-m-2)/3 \rfloor$ disjoint $m \times 3$ zones. Each of these contains at least 2 beaches by Theorem 11, which gives the required bound.

When $n = m$ the westernmost and easternmost $m \times (k+2)$ zones are not disjoint, so the previous argument does not hold. There are at least k beaches in each $2k \times 2k$ corner square and at least $k+1$ in each of the 4 overlapping $(2k+2) \times (2k+2)$ corner squares. Hence each $m \times (2k+2)$ corner zone contains at least $2k+1$ beaches, and since the number of beaches here is even by Theorem 9, it must be at least $2k+2$. If there were only $4k+2$ beaches altogether we would need exactly k beaches in each $2k \times 2k$ corner and another 2 beaches in the 2×2 square in the centre of the chart. But this arrangement would not allow an even number of beaches in each of the lines intersecting this 2×2 square, and so we conclude that the chart contains at least $4k+4 = m+2$ beaches.

When $n = m+1$ a similar argument shows that we certainly need at least $m = 4k+2$ beaches. If there were only this number of beaches altogether we'd have exactly k beaches in each $2k \times 2k$ corner square. By similar considerations to the $n = m$ case we find the remaining 2 beaches must lie in the 2 central cells of the central north-south line, and that the adjacent north-south lines contain no beaches. We consider two cases. If these two beaches both run north-south the cell immediately to their north is a turn. North of this is an odd number of cells which, as in the proof of Theorem 9, must contain an even number of turns and therefore at least one beach.

Suppose instead that the two beaches both run east-west. They may be connected by a pair of turns in the cells immediately to the east (respectively west), but then the two cells to the west (respectively east) contain turns away from the centre. As is the previous case we have an odd number of cells above the centre which must contain another beach.

Thus in either case we get an extra beach giving at least $4k+3$ altogether. Since the total number of beaches must be even we will have at least $4k+4 = m+2$ as required. □

The Table summarises the bounds from Theorems 13, 14 and 15. Recall that we cannot have both m and n odd by Theorem 1. We have only been able to show that these bounds are best possible in 4 of 7 cases. Indeed in the other cases we believe the bounds are not best possible. Note that in each of these m or n is congruent to 2 modulo 4.

Figure 5 shows an 18×26 chart with 28 beaches (compared with the lower bound of 24 from Theorem 15). Note that the western 10 north-south lines contain 10 beaches ($= 2k+2$ in the notation of the proof of Theorem 15) and the next 6 north-

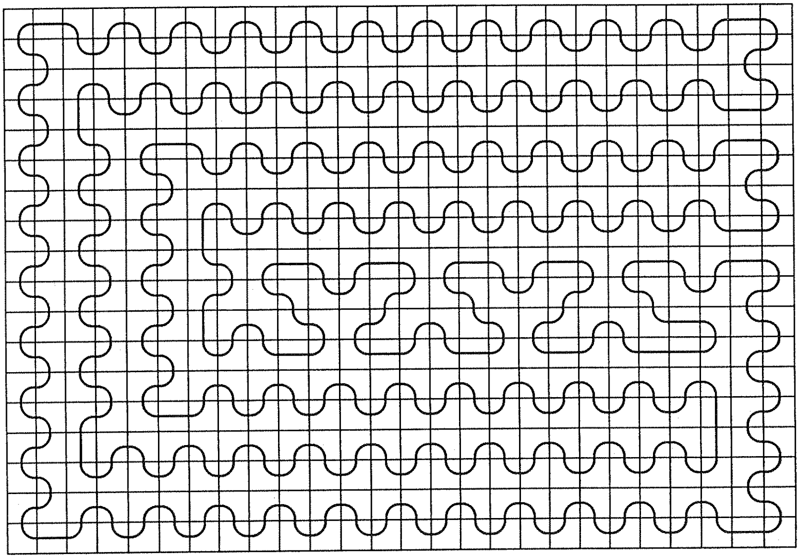

Figure 5: An 18×26 chart with 28 beaches.

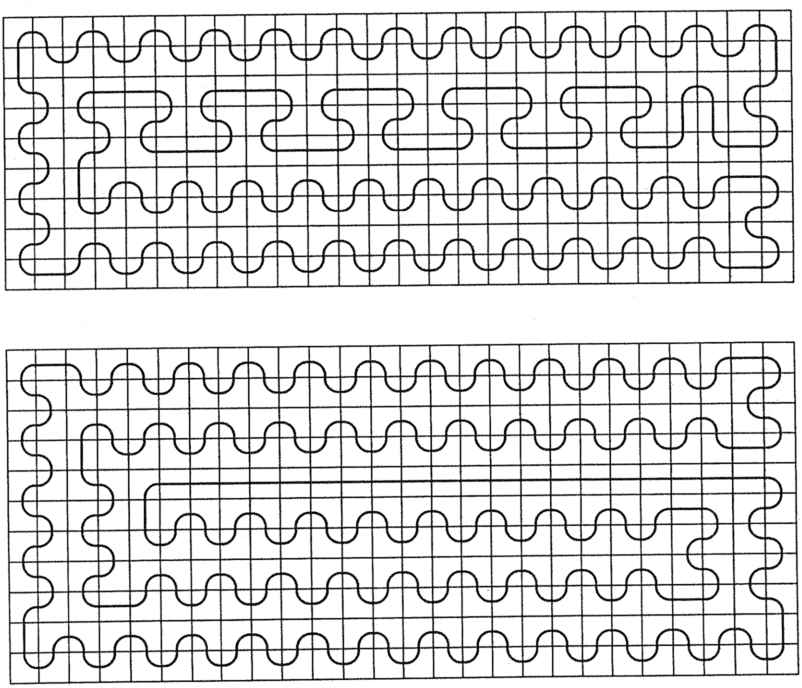

Figure 6: Charts with odd m and even n.

south lines contain 2 disjoint 18×3 zones and 4 beaches, both values predicted by Theorem 15. The final 10 lines contain 14 beaches rather than the predicted 10. The excess is because we have had to extend the central "key pattern" strip to the eastern boundary of the chart. A generalisation of this construction produces $(m+2)/2$ beaches in the western $(m+2)/2$ vertical lines, $2\lfloor(n-(m+2)/2)/3\rfloor$ beaches for the key pattern, $(m-4)/2$ from eastern corner squares and one extra beach to give an even total. This total is then

$$m + 2\lfloor(2n - m - 2)/6\rfloor. \tag{2}$$

This is the best we have been able to do for $m \geq 10$, $m \equiv 2$ $(mod\ 4)$. In the case $m = 6$ the key pattern strip lies on the boundary of the chart which leads, for some values of n, to a slight reduction in the number of beaches given by (2). When $m = 2$ we cannot have fewer than $2n - 4$ beaches.

The final case to consider is when m is odd and $n \equiv 2$ $(mod\ 4)$. Our best constructions for this case are shown in Figure 6 where we have a 9×26 chart and a 11×26 chart, with 28 and 30 beaches respectively. These can be generalised for any $m \equiv 1$ $(mod\ 4)$ or $m \equiv 3$ $(mod\ 4)$ providing $m > 1$. An analysis like that above shows we can construct charts with $n+(m-5)/2$ beaches and $n+(m-3)/2$ beaches as m is congruent to 1 or 3 modulo 4 respectively.

Charts with few Beaches $2 \leq m \leq n$			
m	n	Lower Bound on s	Least s by construction
0 (mod 4)	0 (mod 4)	m	m
0 (mod 4)	2 (mod 4)	m	m
0 (mod 4)	1 (mod 2)	m	m
2 (mod 4)	0 (mod 4)	$m+2+2\lfloor(n-m-2)/3\rfloor$	$m+2\lfloor(2n-m-2)/6\rfloor$
2 (mod 4)	2 (mod 4)	$m+2+2\lfloor(n-m-2)/3\rfloor$	$m+2\lfloor(2n-m-2)/6\rfloor$
2 (mod 4)	1 (mod 2)	$m+2+2\lfloor(n-m-2)/3\rfloor$	$m+2\lfloor(2n-m-2)/6\rfloor$
1 (mod 2)	0 (mod 4)	n	n
1 (mod 4)	2 (mod 4)	n	$n+(m-5)/2$
3 (mod 4)	2 (mod 4)	n	$n+(m-3)/2$

6 Discussion

We have found upper and lower bounds on the number of beaches in an $m \times n$ chart which depend on the residue classes of m and n modulo 4. These lead to bounds on the number of bays and headlands via (1) and Theorem 2. In some cases the bounds are best possible but in others there is a substantial gap between our bounds and our best construction. It would be nice to close this gap.

The sort of questions we have been discussing could also be asked for higher dimensions, or for charts on other surfaces. How would things change if the chart were cylindrical (a pair of opposite boundaries identified) or toroidal (both pairs identified) or spherical (the northern and eastern boudaries identified and the western and southern identified)? A question we have not considered is: How many different ways are there of drawing a coastline in an $m \times n$ chart? This appears to be difficult and has been studied in a disguised setting by Kwong and Rogers [1]. A question similar to those we have been considering appeared recently as a problem in the American Mathematical Monthly [2]. This concerns a path which begins in the northwest quadrate of a chart and ends in the southeast.

7 Acknowledgements

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References

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[2] Fisher, D.C. and Collins, K. L. (proposers), Problem 10406, Amer. Math. Monthly, **101**(1994), 793-794.

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