Some new RBIBDs with block size 5 and PBDs with block sizes ≡ 1 mod 5

R.J.R. Abel
School of Mathematics
University of New South Wales
Kensington, NSW 2033, Australia

M.Greig
5685 Daffodil Drive
West Vancouver
B.C. V7W 1P2, Canada

ABSTRACT

$(v,5,1)$ RBIBDs were known to exist for all but 109 values of $v \equiv 5 \bmod 20$. Solutions are given for all but 6 of these. Also given is a new $(45,5,3)$ RBIBD plus some new incomplete TDs, 14 new $(v,6,1)$ BIBDs and some new PBDs with $K = \{6\} \cup \{\text{Prime powers} \equiv 1 \bmod 5\}$.

1. Introduction

A design is a pair $(X,\mathcal{A})$ where X is a finite set (whose elements are called points) and $\mathcal{A}$ is a collection of subsets of X (called blocks). A group divisible design (GDD), denoted by GD(K, λ, G, v) is a design on v points, divided into groups with sizes from G, and block sizes from K such that two points appear together in λ blocks if in different groups, and in no blocks if in the same group. The parameter λ is called the index of such a design. Also, if $K = \{k\}$ or $G = \{g\}$, we sometimes write k for $\{k\}$ and g for $\{g\}$. A pairwise balanced design (a (v, K, λ) PBD) is a GD$(K, \lambda, 1, v)$; if the block size is uniform, with $K = \{k\}$, then such a PBD is called a balanced incomplete block design, (a (v, k, λ) BIBD). A transversal design, denoted TD(k, v) is a GD$(\{k\}, 1, \{v\}, kv)$.

The notations $B(K)$, $B(k)$ and $T(k)$ respectively are used to represent the sets of v values for which a $(v, K, 1)$ PBD, a $(v, k, 1)$ BIBD and a $TD(k, v)$ exist. The notation $K \cup k^*$ for the block set of a design indicates that one block in the design has size k and all other block sizes come from K; if $k \in K$, then other blocks of size k are possible.

When dealing with GDDs it is sometimes desirable to specify the number of groups of each size. Accordingly, a (K, λ) GDD of type $x^a y^b$ ... is a GDD with index λ, block sizes from K and a groups of size x, b groups of size y, etc.

An incomplete transversal design (an ITD) denoted by $TD(k, v) - TD(k, h)$ is a $TD(k, v)$ with a (sometimes hypothetical) sub-$TD(k, h)$ removed. A list of known ITDs with $v \leq 1000$ and $h \leq 50$ can be found in [2].

A design is called *resolvable* if its blocks can be partitioned into resolution classes where a *resolution class* is a set of blocks containing every point exactly once. Resolvable designs are denoted by the prefix R; also, $RRN(k)$ is used to denote the set of replication numbers for $(v, k, 1)$ RBIBDs, i.e. the set of r values for which $v = (k-1)r + 1 \in RB(k)$. A transversal design is called *idempotent* if it contains at least one resolution class.

A *partial resolution class* in a design is a set of blocks containing no point more than once. Designs with partial resolution classes can sometimes be used to obtain resolvable designs (see for instance, Theorem 5.3).

A useful class of GDDs is *frames*; a GDD is a frame if its blocks can be partitioned into holey resolution classes where a holey resolution class is a set of blocks containing no points from one group and all other points exactly once. It is known that the number of holey resolution classes missing any group of size g is $\lambda g/(k-1)$.

A set K is called PBD-closed if $B(K) = K$. For any k and for any set K, it is known that the sets $B(K)$, $B(k)$ and $RRN(k)$ are PBD-closed [12, 16].

One major aim of this paper is to construct new resolvable BIBDs (RBIBDs) with $k = 5$ and $\lambda = 1$. Simple counting arguments show that if $v \in RB(5)$, then $v \equiv 5 \bmod 20$. Further, there are no v values in this class for which such an RBIBD is known not to exist. In [25] it was noted that these conditions are sufficient, except possibly for the 109 values of v in Table 1.1. (Constructions for $v = 805$, 905 and 1505 were found by Paul Schellenberg and are given in [10]. Alternative constructions for these three values are given in Theorem 5.8.)

Table 1.1

45	105	145	165	185	225	245	285	345	465	525
565	585	645	665	705	765	785	825	885	925	945
1005	1045	1065	1145	1165	1185	1245	1305	1385	1425	1485
1545	1605	1665	1725	1845	1905	1965	2085	2145	2205	2265
2325	2385	2445	2505	2565	2685	2745	2865	2985	3045	3105
3165	3225	3345	3465	3525	3585	3645	3705	3765	3785	3885
3945	4065	4185	4245	4365	4425	4485	4545	4605	4665	4725
4785	4845	4905	4965	5025	5085	5145	5385	5445	5685	5745
5865	5925	5985	6045	6165	6225	6285	6345	6585	6645	6705
6945	7005	7065	7125	7185	7245	7365	7425	7485	7845	

In this paper the number of exceptions in this list is reduced to 6. First, we provide direct constructions for $v = 105$, 165 and give a new direct construction which would appear to work whenever $v = 4p + 1$ and $p \equiv 1 \bmod 10$ is a prime ≥ 61. Later, use is made of known recursive constructions, using these designs as ingredients to obtain several larger $(v, 5, 1)$ RBIBDs. For many of these recursive constructions, some information is needed on PBDs whose block sizes include 6; examples of these designs, including 14 new $(v, 6, 1)$ BIBDs are also provided.

2. A new direct construction

The following theorem summarises the main result of this section:

Theorem 2.1: If $p \equiv 1 \bmod 10$ is prime and $61 \leq p \leq 1151$ then there exists a resolvable $(4p+1, 5, 1)$ BIBD.

Proof: Take the point set as $X = (Z_2 \times Z_2 \times GF(p)) \cup \{\infty\}$. Let x be a primitive root of unity in GF(p), and for $0 \leq t \leq 4$, let C_t be the multiplicative coset consisting of all elements of the form x^{α} with $\alpha \equiv t \bmod 5$. We now construct 3 blocks:

$B_1 = \{(0,0,a_1), (0,0,a_2), (0,0,a_3), (0,1,a_4), (1,0,a_5)\}$
$B_2 = \{(0,0,a_6), (0,1,a_7), (0,1,a_8), (1,0,a_9), (1,0,a_{10})\}$
$B_3 = \{\infty, (0,0,0), (0,1,0), (1,0,0), (1,1,0)\}$

The main problem here is to choose the values $a_1, a_2,\ldots,a_{10}$ so that when we multiply B_1 and B_2 by $(1,1,x^{5t})$ for $0 \leq t \leq (p-11)/10$, we can cycle B_1, B_2 and their multiples mod $(2,2,p)$ and B_3 mod $(-,-,p)$ to produce the required BIBD. The first necessary condition is that for any t, $0 \leq t \leq 4$, and for any $(y,z) \in (Z_2 \times Z_2)$, there is a pair of points $P_1 = (y_1,z_1,a_i)$, $P_2 = (y_2, z_2, a_j)$ both in B_1 or both in B_2 such that $(y_1,z_1) - (y_2,z_2) = (y,z)$ and $a_i - a_j \in C_t$. If this condition is satisfied, then the above procedure will at least give a $(4p+1,5,1)$ BIBD (not necessarily resolvable). For resolvability, we require two extra conditions: (i) If $a \in \{a_1, a_2,\ldots,a_{10}\}$ then so is $-a$ and (ii) For any t, $0 \leq t \leq 4$, exactly two of $a_1, a_2,\ldots,a_{10}$ lie in C_t. If these two conditions hold, then cycling B_1, B_2 and their multiples mod $(2,2,-)$ and combining with B_3 gives a resolution class; cycling these blocks mod p will then produce p resolution classes, completing the resolution.

When $p = 11$, 31 or 41, there are no values of $a_1, a_2,\ldots,a_{10}$ satisfying the above conditions; thus, for these values of p, this approach fails to produce a $(4p+1, 5, 1)$ RBIBD. However, when $p = 41$, a slight modification of this construction works as will be seen in Theorem 2.2; also, for $p = 31$, a (125, 5, 1) RBIBD is easily obtainable from a 3-dimensional affine geometry. The above construction would appear to work whenever $p \geq 61$; in fact the number of non-isomorphic solutions increases very rapidly with p. For $61 \leq p \leq 1151$, Table 2.1 gives a primitive element x in GF(p) plus the values of $v = 4p + 1$ which lie in RB(5) and a possible solution for $a_3, a_4,\ldots,a_{10}$. (For all solutions, $a_1 = 1$ and $a_2 = p-1$.) We simplified the search procedure by applying the extra restrictions that (i) $a_3 < p/2$ and (ii) there was no ± pair in the set $\{a_1, a_3, a_6, a_7, a_9\}$. These restrictions are convenient, as they frequently prevent us searching for partial solutions isomorphic to partial solutions already considered.

Table 2.1

v	p	x	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
245	61	2	6	55	18	7	33	54	43	28
285	71	7	11	60	9	36	29	35	62	42
405	101	2	5	90	82	19	11	96	33	68
525	131	2	5	33	105	98	26	57	74	126
605	151	7	11	140	47	62	78	89	104	73
725	181	2	8	173	96	6	40	175	85	141
765	191	21	2	189	86	44	42	147	105	149
845	211	3	2	112	209	4	25	207	99	186
965	241	7	6	235	62	10	29	212	179	231
1005	251	6	12	117	216	35	9	239	134	242
1085	271	26	2	268	118	3	153	72	199	269
1125	281	3	2	132	279	14	128	267	149	153
1245	311	17	2	33	269	42	130	309	278	181
1325	331	28	7	316	36	15	295	8	323	324
1605	401	3	6	376	19	25	28	373	382	395
1685	421	2	2	403	419	5	18	416	77	344
1725	431	7	20	325	411	29	106	19	412	402
1845	461	3	6	314	449	12	147	455	204	257
1965	491	7	8	233	470	21	22	483	258	469
2085	521	3	2	498	518	3	23	519	252	269
2165	541	2	2	274	539	7	71	534	267	470
2285	571	3	6	545	120	26	45	526	451	565
2405	601	7	2	593	60	8	213	388	541	599
2525	631	3	2	620	115	11	132	499	516	629
2565	641	3	12	594	628	13	47	629	143	498
2645	661	2	4	656	643	5	18	332	329	657

2765	691	3	6	97	689	2	8	685	594	683
2805	701	2	2	698	498	3	203	300	401	699
3005	751	3	7	161	742	9	562	744	590	189
3045	761	6	6	488	755	4	227	757	273	534
3245	811	3	2	809	714	3	97	316	495	808
3285	821	2	6	819	392	2	131	690	429	815
3525	881	3	7	873	717	8	164	646	235	874
3645	911	17	18	906	249	5	268	643	662	893
3765	941	2	2	292	939	3	17	938	649	924
3885	971	6	10	950	709	21	262	612	359	961
3965	991	6	2	153	983	8	239	989	838	752
4085	1021	10	10	588	1011	2	255	1019	433	766
4125	1031	14	6	684	1025	2	292	1029	347	739
4205	1051	7	4	1041	1043	8	10	1047	56	995
4245	1061	2	2	1058	537	3	78	983	524	1059
4365	1091	2	6	962	1085	7	129	937	154	1084
4605	1151	17	2	1149	775	3	360	791	376	1148

Theorem 2.2: There exist (165, 5, 1) and (45, 5, 3) RBIBDs.

Proof: For (165, 5, 1), take the point set as $X = (Z_2 \times Z_2 \times Z_{41}) \cup \{\infty\}$. Consider the following blocks:

B_1 = {(0,0,1), (0,0,40), (0,0,7), (0,1,15), (1,0,6)}
B_2 = {(0,0,2), (0,1,17), (0,1,36), (1,0,5), (1,0,20)}
B_3 = {(0,0,10), (0,0,31), (0,0,34), (1,0,21), (1,1,38)}
B_4 = {(0,0,24), (1,0,3), (1,0,39), (1,1,26), (1,1,35)}
B_5 = {∞, (0,0,0), (0,1,0), (1,0,0), (1,1,0)}

Note that $x = 7$ is a primitive element in Z_{41}. Multiply each of B_1, B_2, B_3, B_4 by $(1,1,y)$ for $y = 1$ and $y = 7^{10} = 9$; then cycle these blocks mod (2,2,–) and combine with B_5. This produces the first resolution class in the BIBD. Finally, cycling mod 41 completes the resolution.

The construction for (45, 5, 3) is similar. Here we take $X = (Z_2 \times Z_2 \times Z_{11}) \cup \{\infty\}$ and use the following base blocks:

$B_1 = \{(0,0,1), (0,0,10), (0,0,7), (0,1,5), (1,1,4)\}$
$B_2 = \{(0,0,3), (0,1,2), (0,1,6), (1,1,8), (1,1,9)\}$
$B_3 = \{(0,0,3), (0,0,8), (0,0,10), (0,1,4), (1,1,1)\}$
$B_4 = \{(0,0,9), (0,1,6), (0,1,7), (1,1,2), (1,1,5)\}$
$B_5 = \{(0,0,5), (0,0,9), (0,0,10), (1,0,6), (1,1,8)\}$
$B_6 = \{(0,0,2), (1,0,4), (1,0,7), (1,1,1), (1,1,3)\}$
$B_7 = \{\infty, (0,0,0), (0,1,0), (1,0,0), (1,1,0)\}$ (3 times)

Cycling any of the following 3 pairs of blocks mod (2,2,–) and combining with B_7 produces a resolution class: (B_1,B_2), (B_3,B_4), (B_5,B_6). Finally cycle mod 11 to complete the resolution.

Theorem 2.3: There exists a (105, 5, 1) RBIBD over $Z_5 \times Z_{21}$.

Proof: Consider the following base blocks:

$B_1 = \{(0,1), (0,13), (0,16), (0,17), (4,3)\}$
$B_2 = \{(0,6), (0,20), (1,10), (2,4), (2,12)\}$
$B_3 = \{(0,8), (1,5), (1,7), (3,9), (3,19)\}$
$B_4 = \{(0,18), (1,14), (2,15), (3,2), (4,11)\}$
$B_5 = \{(0,0), (1,0), (2,0), (3,0), (4,0)\}$
$B_6 = \{(0,0), (1,3), (2,8), (3,14), (4,9)\}$

Cycling each of B_1, B_2, B_3, B_4 mod (5,–) and combining with B_5 produces a resolution class; then cycling mod 21 produces 21 resolution classes. Cycling B_6 mod (5,21) produces 5 resolution classes since all points in B_6 are distinct mod 5.

The next design, although not resolvable itself, will be useful for some recursive RBIBD constructions in Theorem 5.3.

Theorem 2.4: There exists a GD(5,1,9,45) over $Z_5 \times GF(3^2, x^2=x+1)$ whose blocks can be partitioned into 18 partial resolution classes, each of 5 blocks, in such a way that either all or no points from each group appear in each partial resolution class.

Proof: Groups in this GDD are of the form $Z_5 \times \{y\}$ for $y \in GF(3^2)$. Let $B_1 = \{(0,0), (1,1), (1,2), (4,x^2), (4,2x^2)\}$ and $B_2 = \{(0,0), (1,x), (1,2x), (4,x^3), (4,2x^3)\}$. Cycling either B_1 or B_2 mod (5,–) produces a partial resolution class with the required property; cycling mod 3^2 then gives 18 such partial resolution classes.

3. Some New (*v*,6,1) BIBDs

The set RRN(5) is PBD-closed (i.e. if a (*v*, RRN(5), 1) PBD exists, then $v \in$ RRN(5)). Since 6 is the smallest element of RRN(5), it is no surprise that the existence of (*v*,6,1) BIBDs has a major impact on the establishment of the set RRN(5). The following result is given in [3]:

Lemma 3.1: If $v \equiv 1$ or 6 mod 15 then a (*v*, 6, 1) BIBD exists, except when $v \in \{16, 21, 36\}$ and possibly when $v \in \{46, 51, 61, 81, 141, 166, 196, 201, 226, 231, 256, 261, 276, 286, 291, 316, 321, 346, 351, 376, 406, 411, 436, 441, 466, 471, 496, 501, 526, 561, 591, 616, 646, 651, 676, 706, 741, 766, 771, 796, 801, 916, 946, 1096, 1221, 1246, 1251, 1396, 1456, 1486, 1521, 1611, 1671, 1851, 2031\}$.

In this section we obtain 14 new (*v*, 6, 1) BIBDs for *v* = 276, 466, 706, 741, 946, 1096, 1246, 1396, 1456, 1486, 1521, 1611, 1671 and 2031. First however, we need to obtain some new (*v*, $\{6\} \cup f^*$, 1) PBDs and some new incomplete TDs with *k* = 6:

Lemma 3.2: A (*v*, $\{6\} \cup f^*$, 1) PBD exists if either

(i) $v = 5f + 1$ and $4f + 1 \in$ RB(5)

(ii) $v = 6(f - \alpha) + \alpha$ where $\alpha \in \{0, 1, 6\}$, $f - \alpha \in$ T(6) and $f \in$ B(6)

(iii) $f = 16$ and $v \in \{141, 171, 201, 231, 261\}$, or $f = 21$ and $v \in \{166, 196, 226\}$, or $f = 26$ and $v \in \{221, 251, 281\}$, or $f = 31$ and $v \in \{246, 276\}$ or $f = 41$ and $v \in \{266, 296\}$ or $f = 51$ and $v = 346$.

Proof: For (i), construct a (4*f* + 1, 5, 1) RBIBD, and add each of *f* infinite points to a separate resolution class. For (ii), construct a TD(6, *f* – α); then form a (*f*,6,1) BIBD on each of 5 groups plus α infinite points. If $\alpha = 6$, let one block in each (*f*,6,1) BIBD contain the infinite points and delete it. Then form a block of size f on the sixth group plus the infinite points. For $f = 16$ and $v = 141$, see [3]. For the other designs in (iii) except $(v,f) = (266,41)$, we construct a $(v - f, \{5, 6\}, 1)$ PBD over $Z_5 \times Z_{(v-f)/5}$ in such a way that no two points in any base block of size 5 have the same first coordinate mod 5. This ensures that the blocks of size 5 can be partitioned into resolution classes; further, there will always be f such resolution classes. By adding each of f infinite points to a separate resolution class and forming a block on the infinite points, a $(v, \{6\} \cup f^*, 1)$ PBD is obtained. In each case there is one base block of the form {(0,0), (1,0), (2,0), (3,0), (4,0)} which should be developed mod $(-, (v-f)/5)$; the others, which are given below, should be developed mod $(5, (v-f)/5)$.

$(v,f) = (171,16)$:

{(0,0), (0,1), (0,5), (0,14), (0,20), (1,25)}
{(0,0), (0,3), (0,10), (2,14), (2,16), (4,13)}
{(0,0), (0,8), (1,30), (2,26), (3,24), (4,27)}
{(0,0), (1,1), (2,10), (3,23), (4,8)}
{(0,0), (1,2), (2,9), (3,19), (4,14)}
{(0,0), (1,6), (2,20), (3,8), (4,16)}

$(v,f) = (201,16)$:

{(0,17), (0,22), (0,35), (0,14), (0,29), (0,31)}
{(0,0), (1,9), (2,7), (3,1), (4,26)}
{(0,0), (0,1), (1,20), (2,9), (3,13), (4,31)}

Multiply the last two blocks by (1,10) and (1,26) mod (5,37) to produce 4 further base blocks.

$(v,f) = (231,16)$:

{(0,0), (0,1), (0,16), (0,26), (0,40), (1,2)}
{(0,0), (0,2), (0,7), (0,13), (2,3), (2,11)}
{(0,0), (0,20), (1,11), (1,33), (2,14), (4,36)}
{(0,0), (0,9), (1,21), (2,17), (3,33), (4,5)}
{(0,0), (0,12), (1,9), (2,32), (3,30), (4,18)}
{(0,0), (1,6), (2,23), (3,8), (4,7)}
{(0,0), (1,8), (2,26), (3,36), (4,13)}
{(0,0), (1,14), (2,36), (3,19), (4,8)}

(v,f) = (261,16):

{(0,0), (0,1), (0,4), (1,11), (1,24), (3,40)}

{(0,0), (1,29), (2,34), (3,3), (4,11)}

{(0,0), (0,25), (0,42), (1,19), (2,12), (2,28)}

Multiply these blocks by (1,18) and (1,30) mod (5,49) to produce 6 further base blocks.

(v,f) = (166,21):

{(0,0), (0,1), (0,3), (0,19), (0,25), (1,12)}

{(0,0), (1,4), (2,17), (3,9), (4,15)}

{(0,0), (0,8), (0,20), (1,15), (3,14), (3,28)}

{(0,0), (1,10), (2,12), (3,13), (4,2)}

{(0,0), (1,20), (2,25), (3,22), (4,10)}

{(0,0), (1,23), (2,26), (3,5), (4,1)}

(v,f) = (196,21):

{(0,0), (0,1), (0,3), (0,11), (0,15), (1,31)}

{(0,0), (1,4), (2,1), (3,24), (4,22)}

{(0,0), (0,5), (0,18), (1,8), (2,7), (4,34)}

{(0,0), (1,22), (2,5), (3,16), (4,18)}

{(0,0), (0,7), (0,16), (2,30), (3,4), (3,10)}

{(0,0), (1,10), (2,15), (3,9), (4,23)}

{(0,0), (1,7), (2,28), (3,17), (4,9)}

(v,f) = (226,21):

{(0,0), (0,1), (0,14), (0,22), (0,26), (1,4)}

{(0,0), (1,7), (2,15), (3,39), (4,20)}

{(0,0), (0,2), (0,5), (0,11), (2,3), (2,21)}

{(0,0), (1,9), (2,22), (3,34), (4,21)}

{(0,0), (0,7), (1,6), (1,37), (2,11), (4,23)}

{(0,0), (1,10), (2,37), (3,7), (4,2)}

{(0,0), (0,17), (1,14), (2,30), (3,32), (4,24)}

{(0,0), (1,26), (2,17), (3,5), (4,40)}

(v,f) = (221,26):

{(0,0), (0,3), (0,9), (0,11), (0,26), (1,29)}

{(0,0), (1,2), (2,37), (3,36), (4,3)}

{(0,0), (0,14), (0,18), (0,19), (1,12), (3,31)}

{(0,0), (1,5), (2,24), (3,37), (4,14)}

{(0,0), (0,29), (1,17), (1,24), (3,6), (3,33)}

{(0,0), (1,9), (2,10), (3,21), (4,35)}

{(0,0), (1,22), (2,11), (3,3), (4,18)}

{(0,0), (1,30), (2,1), (3,8), (4,16)}

(v,f) = (251,26):

{(0,0), (0,3), (0,27), (0,31), (0,32), (1,36)}

{(0,0), (1,3), (2,11), (3,10), (4,31)}

{(0,0), (0,2), (0,10), (0,36), (1,25), (2,41)}

{(0,0), (1,20), (2,32), (3,22), (4,23)}

{(0,0), (0,15), (0,38), (1,17), (2,27), (3,1)}

{(0,0), (0,12), (1,18), (1,38), (3,36), (3,42)}

{(0,0), (1,39), (2,1), (3,32), (4,17)}

{(0,0), (1,43), (2,25), (3,17), (4,13)}

{(0,0), (1,42), (2,26), (3,37), (4,32)}

$(v,f) = (281,26)$:

{(0,0), (0,3), (0,5), (0,24), (0,25), (1,6)}

{(0,0), (0,4), (0,15), (0,38), (1,8), (2,39)}

{(0,0), (0,9), (0,44), (1,36), (2,20), (3,46)}

{(0,0), (0,6), (0,43), (1,15), (3,9), (3,48)}

{(0,0), (0,18), (1,30), (1,40), (2,17), (3,11)}

{(0,0), (1,25), (2,38), (3,33), (4,2)}

{(0,0), (1,41), (2,19), (3,15), (4,49)}

{(0,0), (1,50), (2,47), (3,20), (4,37)}

{(0,0), (1,16), (2,4), (3,22), (4,41)}

{(0,0), (1,11), (2,2), (3,39), (4,46)}

$(v,f) = (246,31)$:

{(0,0), (0,1), (0,8), (0,10), (0,21), (0,39)}

{(0,20), (0,23), (1,3), (1,40), (4,14), (4,29)}

{(0,8), (0,35), (2,13), (2,30), (3,12), (3,31)}

{(0,0), (1,2), (2,16), (3,35), (4,3)}

{(0,0), (1,4), (2,31), (3,41), (4,13)}

{(0,0), (1,5), (2,36), (3,29), (4,8)}

Multiply each of the blocks in the right-hand column by (1,−1) mod (5,43) to produce 3 further base blocks.

$(v,f) = (276,31)$:

{(0,0), (0,1), (0,8), (0,10), (0,23), (1,45)}

{(0,0), (0,4), (0,16), (0,35), (2,11), (2,40)}

{(0,0), (0,6), (0,11), (1,32), (2,3), (4,1)}

{(0,0), (0,3), (0,24), (1,9), (1,41), (3,43)}

{(0,0), (0,1), (2,13), (3,29), (4,31)}

{(0,0), (1,3), (2,14), (3,41), (4,35)}

{(0,0), (1,4), (2,27), (3,14), (4,21)}

{(0,0), (1,8), (2,33), (3,23), (4,7)}

{(0,0), (1,13), (2,42), (3,12), (4,9)}

{(0,0), (1,15), (2,39), (3,37), (4,19)}

$(v,f) = (296,41)$:

{(0,0), (0,3), (0,12), (0,17), (0,25), (0,35)}

{(0,0), (0,6), (0,30), (0,50), (1,16), (2,13)}

{(0,0), (0,4), (0,15), (1,3), (2,45), (2,47)}

{(0,0), (1,12), (2,44), (3,18), (4,2)}

{(0,0), (1,21), (2,29), (3,48), (4,50)}

{(0,0), (1,5), (2,25), (3,47), (4,10)}

{(0,0), (1,18), (2,49), (3,36), (4,21)}

{(0,0), (1,29), (2,40), (3,49), (4,24)}

{(0,0), (1,15), (2,11), (3,39), (4,28)}

{(0,0), (1,6), (2,39), (3,43), (4,38)}

{(0,0), (1,34), (2,26), (3,50), (4,6)}

$(v,f) = (346,51)$:

{(0,0), (0,1), (0,7), (0,21), (0,34), (0,36)}
{(0,0), (0,3), (0,8), (0,19), (0,50), (1,42)}
{(0,0), (0,4), (0,22), (1,44), (3,45), (3,55)}
{(0,0), (1,1), (2,37), (3,16), (4,22)}
{(0,0), (1,2), (2,45), (3,31), (4,7)}
{(0,0), (1,3), (2,51), (3,49), (4,1)}
{(0,0), (1,4), (2,13), (3,37), (4,32)}
{(0,0), (1,5), (2,12), (3,3), (4,44)}
{(0,0), (1,8), (2,55), (3,42), (4,3)}
{(0,0), (1,10), (2,35), (3,9), (4,28)}
{(0,0), (1,12), (2,40), (3,36), (4,6)}
{(0,0), (1,13), (2,27), (3,43), (4,10)}
{(0,0), (1,17), (2,47), (3,6), (4,27)}

For $(v,f) = (266,41)$, we work over $Z_5 \times Z_3 \times Z_{15}$. There is one short block, {(0,0,0), (0,0,3), (0,0,6), (0,0,9), (0,0,12)} which generates a resolution class on the non-infinite points by adding each of the triples (x,y,z) for $x \in Z_5$, $y \in Z_3$ and $z \in \{0,1,2\}$. The other 10 blocks given below should be developed mod (5,3,15). As before, no 2 points in any base block of size 5 have the same first coordinate mod 5; this ensures that the blocks of size 5 each generate 5 resolution classes on the non-infinite points.

{(0,0,0), (0,0,1), (0,0,11), (0,1,4), (0,2,14), (1,0,0)}
{(0,0,0), (0,1,6), (0,1,14), (0,2,4), (0,2,6), (2,0,0)}
{(0,0,0), (1,0,6), (2,2,4), (3,0,4), (4,2,11)}
{(0,0,0), (1,0,7), (2,2,6), (3,0,13), (4,2,7)}
{(0,0,0), (1,2,3), (2,1,7), (3,2,13), (4,1,10)}
{(0,0,0), (1,0,13), (2,1,12), (3,0,3), (4,0,4)}
{(0,0,0), (1,2,8), (2,1,3), (3,1,5), (4,0,5)}
{(0,0,0), (1,0,12), (2,2,13), (3,0,1), (4,1,13)}
{(0,0,0), (1,0,5), (2,0,8), (3,1,10), (4,2,4)}
{(0,0,0), (1,1,10), (2,2,8), (3,2,1), (4,0,6)}

Theorem 3.3: Let $v = (k-2)h + 1$. If $k - 1$ is a prime power, and there exist both a $TD(k, v+h) - TD(k, h)$ and an idempotent $TD(k, v)$, then there exists a $TD(k, kv-1) - TD(k, (k-2)h)$.

Proof: The proof of this theorem can be divided into four steps. Thus:

(i) The point set for TD(k, $kv-1$) - TD(k, $(k-2)h$) will be taken as $I_k \times [(I_v \times I_{k-1}) \cup (H \times I_{k-2})]$ where for any x, I_x denotes a set of size x, and H has size h. Groups will consist of points with the same values from I_k, and the hole set will be taken as $I_k \times (H \times I_{k-2})$. Let D be a TD($k$, $v+h$) - TD(k,h) on $I_k \times [I_v \cup H]$, the hole set being $I_k \times H$. The design D contains v blocks with no points from $I_k \times H$; further, simple counting shows every point in $I_k \times I_v$ appears in exactly one of these blocks.

(ii) If B is a block in D containing no point from $I_k \times H$, then construct a TD(k, $k-1$) on $I_k \times B$.

(iii) If B is a block in D containing one point, P, from $I_k \times H$, then construct a resolvable TD($k-1$, $k-1$) on $(B \backslash P) \times I_{k-1}$ in such a way that one resolution class consists of the $k-1$ blocks $(B \backslash P) \times \{y\}$, for $y \in I_{k-1}$. There remain $k-2$ resolution classes in this design; add each point from $P \times I_{k-2}$ to one of them.

(iv) Two points not both in the hole set $I_k \times (H \times I_{k-2})$ will appear in a block from (ii) or (iii) unless they both have equal coordinates from I_{k-1} and unequal coordinates from I_v. So for each $y \in I_{k-1}$, we construct a TD(k, v) - v·TD(k, 1) on $I_k \times [I_v \times \{y\}]$, the v holes of size 1 being on $I_k \times \{z\} \times \{y\}$, for $z \in I_v$.

The blocks in (i), (ii), (iii) and (iv) give the required TD(k, $kv - 1$) - TD(k, $(k-2)h$).

Remark 3.4: Examples of new ITDs obtainable from Theorem 3.3 are TD(6,77) - TD(6,12) and TD(6,101) - TD(6,16). These two designs are used later in Tables 3.1 and 5.1 to obtain a (466,6,1) BIBD and a (2445,5,1) RBIBD respectively.

We now give a few more ITDs using quasi-difference matrices (QDMs). A $(z,k,\lambda_1,\lambda_2;h)$ QDM over an abelian group G of size v is an array $Q = (q_{i,j})$ with k rows and $\lambda_1(z-1+2h) + \lambda_2$ such that (i) each entry is either blank (denoted by –) or contains an element of G, (ii) for any two rows i,j the multi-set of differences $q_{i,l} - q_{j,l}$ with $q_{i,l}$, $q_{j,l}$

both not empty contains each non-zero element of G λ_1 times and zero λ_2 times. If $\lambda_1 = \lambda_2 = 1$, then existence of such a QDM implies existence of a TD(k,$z+h$) - TD(k,h).

Theorem 3.5: If $(v,h) \in \{(20,4), (30,6), (31,5), (56,11)\}$ then a $(v-h,k,1,1;h)$ QDM and a TD(6,v) - TD(6,h) both exist.

Proof: For $(v,h) = (20,4)$ consider the following array over GF(16, $x^4 = x+1$):

A =

x^{13}	x^{14}	x^2	—	0	1
x^2	x^{13}	x^{14}	1	—	0
x^{14}	x^2	x^{13}	0	1	—
—	0	x	x^{13}	x	x^8
x	—	0	x^8	x^{13}	x
0	x	—	x	x^8	x^{13}

Replace each column $(a,b,c,d,e,f)^T$ of A by the four columns $(a+w,\ b+w{\cdot}x^5,\ c+w{\cdot}x^{10},\ d+w{\cdot}x,\ e+w{\cdot}x^6,\ f+w{\cdot}x^{11})^T$ for $w = 0,\ 1,\ x^5$ and x^{10}. This gives a (16,6,1,1;4) QDM.

For the other 3 cases, we give 2 arrays A_1, A_2 and define an automorphism T by $T(a,b,c,d,e,f)^T = (b,c,d,e,a,f)^T$; the required $(v-h,6,1,1;h)$ QDM is then obtained by applying the automorphism group of order 5 generated by T to the columns of A_1 and then appending the columns of A_2.

$(v,h) = (30,6)$:

A_1 =

0	0	0	0	0	0	0
15	7	19	8	2	12	18
19	13	5	11	18	23	8
8	22	2	12	23	21	4
1	—	—	—	—	—	—
—	3	4	6	10	12	1

A_2 =

0
0
0
0
0
—

$(v,h) = (31,5)$:

$A_1 =$	0	0	0	0	0	0	0	$A_2 =$	0
	5	9	25	18	12	4	10		0
	7	23	19	21	8	1	16		0
	20	8	12	16	15	9	14		0
	9	—	—	—	—	10	—		0
	—	3	17	2	16	23	25		0

$(v,h) = (56,11)$:

$A_1 =$	0	0	—	—	—	—	—	—	—	—	—	—	—	$A_2 =$	0	0
	32	12	0	0	0	0	0	0	0	0	0	0	0		0	9
	35	1	4	22	16	42	15	27	26	37	10	38	30		0	18
	43	37	9	33	14	21	36	10	12	38	30	26	20		0	27
	5	6	28	17	13	1	38	23	29	11	8	32	16		0	36
	—	—	15	29	44	40	16	37	17	13	34	35	38		0	—

Remark 3.6: For two of the TD(6,v) - TD(6,h)'s in the previous theorem, namely for (v,h) = (20,4) and (30,6) we have $v = 5h$. TD(6,5h) - TD(6,h)'s can be used to construct (5,1) frames of type $[4h]^6$. (See Theorem 5.5). Also the last design with (v,h) = (56,11) can be used to obtain a (5,1) GDD of type 6^{45}. (See [19] and [22] for further details.)

For some of our recursive design constructions, we make use of Wilson's fundamental GDD construction [21]. Below is a brief description of this construction together with its extension to frame constructions [20]:

Theorem 3.7: Suppose that D is a master GD(K_1, λ_1, M, v) with groups G_j ($j = 1,\dots,g$) and each point x in D is assigned a non-negative weight $w(x)$. Then:

(i) If for each block $B = \{x_1, x_2,\dots,x_k\}$ in D there exists a (K_2, λ_2) GDD of type ($w(x_1)$, $w(x_2)$,...,$w(x_k)$), then there exists a (K_2, $\lambda_1 \cdot \lambda_2$) GDD with group type $(\sum_{x \in G_j} w(x): j = 1,\dots,g)$.

(ii) If $\lambda_1 = 1$ and for each block $B = \{x_1, x_2,...,x_k\}$ in D there exists a (K_2, λ_2) frame of type $(w(x_1), w(x_2),...,w(x_k))$, then there exists a (K_2, λ_2) frame of type $(\sum_{x \in G_j} w(x): j = 1,...g)$.

The main result of this section now follows: 14 new (v, 6, 1) BIBDs are given. Ten of these are obtained by the following construction sometimes known as singular indirect product (SIP):

Theorem 3.8: Let $v = k(w-f) + f + (k-1)a$ where $0 \leq a \leq f$. Suppose that the following designs exist: an $(f+(k-1)a, K, 1)$ PBD, a $(w, K \cup f^*, 1)$ PBD, and a TD$(k, w-f+a)$ - TD$(6,a)$. Then a $(v, K \cup \{k\} \cup (f+(k-1)a)^*, 1)$ PBD exists. If further, $f + (k-1)a \in B(K \cup \{k\})$, then $v \in B(K \cup \{k\})$.

Proof: See [13] or [24].

Remark 3.9: When using Theorem 3.8, we generally do not indicate the construction for the required TD$(6, w-f+a)$ - TD$(6, a)$. A table of known TD$(6, x)$ - TD$(6, y)$'s for $x \leq 1000$, $y \leq 50$ can be found in [2]. For details of the construction methods see [6] and [9].

Theorem 3.10: If $v \in \{276, 466, 706, 741, 946, 1096, 1246, 1396, 1456, 1486, 1521\}$, then $v \in B(6)$.

Proof: For $v = 276$, construct a $(276, \{6\} \cup 31^*, 1)$ PBD as in Lemma 3.2; then delete the big block and form a (31,6,1) BIBD on the points in it. For all the other values of v, use Theorem 3.8 with $K = \{6\}$, $k = 6$ and the values of w, f, a shown in Table 3.1. For the required $(w, \{6\} \cup f^*, 1)$ PBDs see Lemma 3.2; also, from [2], we know that all the required TD$(6, w-f+a)$ - TD$(6, a)$'s exist. For $v = 466$, the required TD(6, 77) - TD(6, 12) is also obtainable from Theorem 3.3.

Table 3.1

v	w	f	a	$f+(k-1)a$	v	w	f	a	$f+(k-1)a$
466	81	16	12	76	1246	221	26	10	76
706	131	26	10	76	1396	246	31	15	106
741	131	26	17	111	1456	251	26	16	106
946	166	21	11	76	1486	251	26	22	136
1096	206	41	13	106	1521	266	41	26	171

Theorem 3.11: $\{1611, 1671, 2031\} \subset B(6)$.

Proof: These constructions are obtained using Theorem 3.7. For $v = 1611$ and 1671, take a TD(18,17), give weight 20 to 4 or 8 points in the last group and weight 5 to all other points. Since there exist (6,1) GDDs of types 5^{18} and $5^{17}20^{1}$ (by deleting one point from a (91,6,1) BIBD or adding 20 infinite points to separate resolution classes in an (85,5,1) RBIBD), Theorem 3.7 can be used to obtain (6,1) GDDs of types $85^{17}145^{1}$ and $85^{17}205^{1}$. Now add 21 infinite points; form a (106,6,1) BIBD on one group of size 85 plus the infinite points and form a $(v, \{6\} \cup 21^*, 1)$ PBD for $v = 106$, 166 or 226 on each other group plus the infinite points. In each case let the block of size 21 contain the infinite points and delete it. These $(v, \{6\} \cup 21^*, 1)$ PBDs come from Lemma 3.2(i) or (iii). For $v = 2031$, the construction is similar, but here we start with a (6,1) GDD of type 5^{13} and inflate it using TD(6,31) to obtain a (6,1) GDD of type 155^{13}; we then add 16 infinite points and form a (171,6,1) BIBD on one group plus the infinite points and a $(171, \{6\} \cup 16^*, 1)$ PBD on each other group plus the infinite points. Again each block of size 16 should contain the infinite points and should be deleted.

4. Some New $(v, Q, 1)$ PBDs for $Q = \{6\} \cup \{\text{Prime powers} \equiv 1 \bmod 5\}$

The existence of these PBDs was investigated in [14]. A few improvements were stated, but not proved, in [4]. In this section, we prove these improvements and obtain a few more.

Theorem 4.1 [14]: Let $Q = \{6\} \cup \{\text{Prime powers} \equiv 1 \bmod 5\}$. If $v \equiv 1 \bmod 5$, then $v \in B(Q)$ except possibly for $v \in E_1 \cup E_2$ where $E_1 = \{21, 26, 36, 46, 51, 56, 86, 116, 146, 166, 196, 221, 226, 236, 286, 291, 316, 321, 326, 351, 386, 411, 416, 441, 446, 471, 476, 501, 536, 566, 596, 626, 651, 686, 716, 746, 771, 776, 801, 806, 866, 896, 926, 986, 1046\}$ and $E_2 = \{141, 161, 171, 201, 206, 231, 261, 266, 276, 296, 336, 356, 376, 561, 591, 621, 706, 711, 741, 766, 831, 946, 956, 1016, 1076, 1106, 1121, 1156, 1196\}$.

We now prove:

Theorem 4.2: If $v \notin E_1$ and $v \equiv 1 \bmod 5$, then $v \in B(Q)$.

Proof: Because of Theorem 4.1, we only need to consider the values of v in E_2. $\{171, 336, 621, 706, 711, 741, 831, 946, 1156\} \subset B(6) \subset B(Q)$ (see Theorem 3.10 for $v = 276, 706, 741, 946$ and [3] for the rest). By Lemma 3.2(iii), $\{141, 201, 231, 261\} \subset B(\{6, 16\}) \subset B(Q)$ and $\{266, 296\} \subset B(\{6, 41\} \subset B(Q)$. Lemma 3.2(i) gives $206 \in B(\{6, 41\})$, $356 \in B(\{6, 71\})$ and $956 \in B(\{6, 191\})$; hence $\{206, 356, 956\} \subset B(Q)$. For $t = 1016$, take TD(7,7) as the master GDD in Theorem 3.7; then give weight 20 to all points in the first 6 groups and weight 25 to all points in the last group. Since (6,1) GDDs of types 20^6 and $20^6 25^1$ exist ([1], [15]), this gives a (6,1) GDD of type $140^6 175^1$. Then form a (141, {6, 16}, 1) PBD or (176, {11, 16}, 1) PBD on each group plus an infinite point. The first of these comes from Lemma 3.2(iii) and the second exists since $176 = 11 \cdot 16$. Finally, for $v = 161, 376, 561, 591, 766, 1076, 1106, 1121, 1196$, we can use the SIP construction in Theorem 3.8 with $K = Q$, $k = 6$, and $(w,f,a) = (31,6,1)$, (66,6,2), (96,16,13), (106,21,12), (141,16,0), (181,31,29), (186,31,29), (201,16,5), (206,41,33) respectively. The required $(w, \{6\} \cup f^*, 1)$ PBDs all come from Lemma 3.2.

Remark 4.3: If K is a PBD-closed set and S is a subset of K such that $B(S) = K$, then S is called a *generating set* for K. If $x \in K$ and $x \notin B(K \setminus \{x\})$, then x is said to be *essential* in K. In [11], it was noted that for $K = \{v\colon v \equiv 1 \bmod 5\}$, a generating set is $S = \{6, 11, 16, 21, 26, 36, 41, 46, 51, 56, 61, 71, 86, 101, 116, 131, 146, 166, 191, 196, 221, 226, 231, 236, 251, 261, 266, 286, 291, 296, 311, 316, 321, 326, 351, 356\}$. In [11], it was also noted that values ≤ 41 in S are essential in K, and that it was not known whether the values ≥ 46 in S are essential in K. However, Lemma 3.2(i) with $f = 26, 71$ shows that 131, 356 are inessential, and Lemma 3.2(iii) shows that 166, 196, 221, 226, 231, 251, 261, 266 and 296 are inessential. Adding an infinite point to the groups of the (6,1) GDD of type $(20^6)(25^1)$ in [15] gives $146 \in B(\{6, 21, 26\})$, so 146 is also inessential. By Lemma 3.2(ii), $66 \in B(\{6,11\})$; hence by Theorem 3.8, $351 = 6(66-11) + 21 \in B(\{6, 11, 21\})$, so 351 is inessential. Finally, by Lemma 3.1 there exists a (66, 6, 1) BIBD and hence also (6,1) GDDs of types 5^{13}; inflating this GDD by 5 using Theorem 3.7 gives a (6,1) GDD of type 25^{13}. Adding an infinite point to the groups gives $326 \subset B(\{6, 26\})$. Hence 326 is inessential.

5. Recursive Constructions for (*v*,5,1) RBIBDs

There are several known recursive constructions for $(v,5,1)$ RBIBDs. The first one in Lemma 5.1 is Harrison's Theorem and is proved in [17].

Lemma 5.1: If $\{km, kn\} \subset RB(k)$ and a $TD(k+1, n)$ exists, then $kmn \in RB(k)$.

Corollary 5.2: $\{825, 1425, 2145, 3465, 7125\} \subset RB(5)$.

Proof: Use Lemma 5.1 ($825 = 5 \cdot 5 \cdot 33$, $1425 = 5 \cdot 5 \cdot 57$, $2145 = 5 \cdot 13 \cdot 33$, $3465 = 5 \cdot 21 \cdot 33$, $7125 = 5 \cdot 25 \cdot 57$). The required $(v,5,1)$ RBIBDs for $v = 105$, 165 and 285 come from Theorems 2.1 – 2.3.

In some cases one of the designs in Lemma 5.1 need not be resolvable, provided its blocks can be partitioned into a sufficiently small number of partial resolution classes. The following theorem is a special case of construction 4.4 in [18]; an outline of the proof can also be found in [10].

Theorem 5.3: Suppose the following designs exist: a resolvable $(k,1)$ GDD of type g^u and a $(k,1)$ GDD of type g^t whose blocks can be partitioned into s partial resolution classes. Further, suppose that whenever any point P appears in any one of these partial resolution classes, then so does every point in the same group as P. Let

$r_u = g(u-1)/(k-1)$ and $r_t = g(t-1)/(k-1)$. Then if $r_u \leq s - r_t$, there exists a resolvable $(k,1)$ GDD of type g^{tu}.

Corollary 5.4: If $v \in \{945, 1485, 2385, 2745, 4725\}$, then $v \in$ RB(5).

Proof: Use Theorem 5.3 with $k = g = 5$ and $t = 9$. The required v values equal $5 \cdot 9 \cdot u$ for $u = 21, 33, 53, 61, 105$; also, the required (5,1) GDD of type 5^9 is given in Theorem 2.4. Therefore, Theorem 5.3 gives a resolvable (5,1) GDD of type $5^{v/5}$. Note that $r_t = 10$, $s = 18$, and in all cases $r_u = 5(u-1)/4 \geq s - r_t = 8$. Filling in each group then gives a $(v,5,1)$ RBIBD.

The next two lemmas provide a method of constructing frames and a method of obtaining RBIBDs from such frames.

Lemma 5.5 [19]: If a TD$(k+1,kw)$ - TD$(k+1,w)$ exists, then a $(k,1)$ frame of type $((k\text{-}1)\,w)^{(k+1)}$ also exists.

Lemma 5.6 [10]: Suppose a $(k,1)$ frame of type $(g_1, g_2,\ldots,g_t)$ exists, $h \in \{1, k\}$, and

$g_i + h \in$ RB(k) for all $i \in \{1, 2,\ldots,t\}$. Then $v = (\sum_{i=1}^{t} g_i) + h \in$ RB(k).

Theorem 5.7: $\{145, 1145, 1165\} \subset$ RB(5).

Proof: For $v = 145$, a (5,1) frame of type 24^6 exists by Lemma 5.5 (see Theorem 3.5 for the required TD(6,30) - TD(6,6)). Now use Lemma 5.6 with $h = 1$. For $v = 1145$, 1165 respectively, start with master (6,1) GDDs of types 5^{19} and $20^6 25^1$. By Lemma 5.5, there exist (5,1) frames of types 12^6 and 8^6, since TD(6,15) - TD(6,3) and TD(6,10) - TD(6,2) both exist (see [8] and [5]). Using these frames as ingredients in Theorem 3.7 we obtain (5,1) frames of types 60^{19} and $160^6 200^1$. Since $\{65, 165, 205\} \subset$ RB(5), the results now follow from Theorem 5.6 with $h = 5$.

The remaining recursive constructions in this section make use of the fact that RRN(5) is PBD-closed. By Theorem 4.2 we have 376 $\in$ B({6, 16}); in addition, {146, 326} $\subset$ B({6, 21, 26}) by Remark 4.3 and {141, 166, 196, 201, 221, 226, 231, 261, 266, 296, 346} $\subset$ B({6, 16, 21, 26, 41, 51}) by Lemma 3.2. Further, 476 $\in$ B({6, 26}): start with a (6,1) GDD of type 5^{19}, inflate by 5 using Theorem 3.7 to obtain a (6,1) GDD of type 25^{19} and add an infinite point to the groups. Hence, by PBD-closure of RRN(5) we have:

Theorem 5.8: {141, 146, 166, 196, 201, 221, 226, 231, 261, 266, 296, 326, 346, 376, 476} $\subset$ RRN(5), i.e. {565, 585, 665, 785, 805, 885, 905, 925, 1045, 1065, 1185, 1305, 1385, 1505, 1905} $\subset$ RB(5).

We now prove:

Theorem 5.9: If $v \in$ {705, 1545, 2265, 2505, 2865, 2985, 3105, 3225, 3345, 3585, 3705, 3945, 4065, 4425, 4665, 4785}, then $v \in$ RB(5).

Proof: These follow from the SIP construction in Theorem 3.8. For the given values of v, $r = (v - 1)/4$ can be written as $6(w - 6) + (26 = 6 + 5 \cdot 4)$ where $w \in$ {31, 66, 96, 106, 121, 126, 131, 136, 141, 151, 156, 166, 171, 186, 196, 201}; thus SIP together with PBD-closure of RRN(5) gives $r \in$ RRN(5) and hence, $v \in$ RB(5). The required (w, RRN(5) $\cup$ 6*, 1) PBDs are obtainable by Lemma 3.2(i) for w = 131, 206, by Lemma 3.2(iii) for w = 141, 166, 196, 201 and by Lemma 3.1 for the rest.

Table 5.1 gives some other new elements of RRN(5) obtained by SIP (with K = RRN(5), k = 6) together with the appropriate values of w,f,a used. It also gives the values of $v = 4r + 1$ which consequently lie in RB(5). As before, the required (w, RRN(5) $\cup$ f*,1) PBDs come from Lemma 3.2 and for the required TD(6,w–f+a) – TD(6,a)'s, see [2].

Table 5.1

v	*r*	*w*	*f*	*a*	*f*+5*a*
1665	416	81	16	2	26
2205	551	106	21	4	41
2325	581	106	21	10	71
2445	611	106	21	16	101
2685	671	131	26	3	41
3165	791	156	31	2	41
3785	946	166	21	11	76
4185	1046	181	31	23	146
4485	1121	196	21	16	101
4545	1136	206	41	21	146
4845	1211	221	26	3	41

We now set about constructing the larger unknown (v, 5, 1) RBIBDs, (i.e. $v \geq 4905$). Theorem 5.10 below can be used for all of these (except $v = 7125$, which was obtained in Corollary 5.2). This theorem uses Wilson's Fundamental GDD Construction (Theorem 3.7) and was proved in [23]. It makes use of ({6, 16}, 1) GDDs of types 5^{15}, 5^{16}, $5^{15}15^1$ and $5^{16}15^1$. For the first two, delete one point from a (76, 6, 1) BIBD or one point not in the large block of an (81, $\{6\} \cup 16^*$, 1) PBD. Then use the deleted point to define groups. For the last two, (96, {6, 16}, 1) and (91, {6, 16}, 1) PBDs can be obtained by filling in the groups of a TD(6, 16) or by filling in the groups of a TD(6, 15) with an infinite point; delete a non-infinite point and use it to define groups.

Theorem 5.10: Suppose $r = 75t + 5u + 15w + \alpha$, where $0 \leq u \leq t$, $0 \leq w \leq t$, $\alpha \in \{1,6\}$, and TD(17, t) exists. Then:

(i) If $\alpha = 1$ and $\{5t + 1, 5u + 1, 15w + 1\} \subset$ RRN(5), then $r \in$ RRN(5).

(ii) If $\alpha = 6$, $\{5t + 6, 15w + 6\} \subset$ B(RRN(5) $\cup$ 6^*), and $5u + 6 \in$ RRN(5), then $r \in$ RRN(5).

Table 5.2 gives several applications of Theorem 5.10. In all cases, we have $5u + \alpha \in \{26, 41, 71\}$; thus $5u + \alpha \in$ RRN(5) by Theorems 2.1 – 2.3. Existence of the required (q, RRN(5) $\cup \alpha^*$, 1) PBDs for $q \in \{5t + \alpha, 15w + \alpha\}$ follows from Lemmas 3.1, 3.2 and Theorem 4.2.

Table 5.2

	v	r
$t = 16$, $u = 5$, $\alpha = 1$	4905, 4965, 5025	1226, 1241, 1256
$t = 16$, $u = 14$, $\alpha = 1$	5085, 5145, 5385, 5445	1271, 1286, 1346, 1361
$t = 17$, $u = 13$, $\alpha = 6$	5685, 5745, 5865, 5925, 5985, 6045, 6165	1421, 1436, 1466, 1481, 1496, 1511, 1541
$t = 19$, $u = 14$, $\alpha = 1$	6225, 6285, 6345, 6585, 6645, 6705, 6945	1556, 1571, 1586, 1646, 1661, 1676, 1736
$t = 23$, $u = 4$, $\alpha = 6$	7005, 7245	1751, 1811
$t = 23$, $u = 7$, $\alpha = 6$	7065, 7365	1766, 1841
$t = 23$, $u = 13$, $\alpha = 6$	7185, 7425, 7485, 7845	1796, 1856, 1871, 1961

We can now summarise the results of Sections 2 and 5. Here, constructions have been provided for 103 of the 109 previously unknown (v,5,1) RBIBDs mentioned in Table 1.1. As a result we have the following theorem:

Theorem 5.11: A resolvable (v, 5, 1) BIBD exists for all $v \equiv 5 \bmod 20$, except possibly for the 6 values of v in Table 5.3.

Table 5.3

45	185	225	345	465	645

Additional note:

Recently the first author found a new TD(7,54) - TD(7,9). By Theorem 5.5, this implies existence of a (6,1) GDD of type 45^7; adding an infinite point to the groups of this design gives $316 \in B(\{6, 46\})$. Thus, 316 is also an inessential element in the set K in Remark 4.3.

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