THE MATCHING PROPERTY OF NESTED SETS

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Abstract

The matching property of nested sets of pairs on [2n] is introduced. Various structural properties of the matching nested sets are presented and the relation between matching nested sets and planar permutations is exhibited.

1. Introduction

A set S of pairwise disjoint pairs of $[2n]=\{1,2,\ldots,2n\}$ such that $\bigcup_{\{\alpha,b\}\in S}\{\alpha,b\}=[2n]$ and for any $\{\alpha,b\}, \{c,d\}\in S$ we never have $\alpha<c<b<d$, is called *nested* set of pairs on [2n].

It is clear that the nested set is a special case of the non-crossing partition [1], with blocks containing exactly two elements. It is well known that the nested sets are related to nested parentheses [3] and are used in the study of planar permutations [4] and Jordan sequences [2].

The structure of the set N_{2n} of nested sets of pairs on [2n] is studied in [5]. In this paper, we introduce and study the matching property in the set N_{2n}, which allows two nested sets to be joined in such a way that a unique cycle on [2n] is obtained. We show how the matching nested pairs are generated, using pairs of smaller size.

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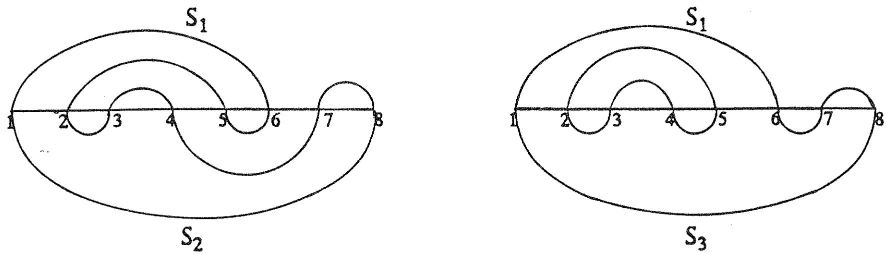
2. Matching Property

Let $S \in N_{2n}$ and $A \subseteq S$; we write $I(A) = \bigcup_{\alpha \in A} \alpha$.

We say that two sets S_1, $S_2 \in N_{2n}$ are *matching* iff $I(A)=I(B)$, $A \subseteq S_1$, $B \subseteq S_2$ imply that $I(A)=\varnothing$ or $[2n]$.

So, the sets $S_1=\{\{1,6\},\{3,4\},\{5,2\},\{7,8\}\}$, $S_2=\{\{1,8\},\{3,2\},\{5,6\},\{7,4\}\}$ are matching, whereas the sets S_1 and $S_3=\{\{1,8\},\{3,2\},\{5,4\},\{7,6\}\}$ are not, as it is indicated geometrically by the cycles of Fig. 1.



Matching nested sets

Non matching nested sets

Fig. 1

Now, given an $S \in N_{2n}$ and $c \in Z$, we consider the *translation* $S+c= \{\{\alpha+c, b+c\} : \{\alpha, b\} \in S\}$, where all numbers are taken mod 2n, so that $S+c \in N_{2n}$. An equivalence relation $\sim$ on N_{2n} is defined by $S_1 \sim S_2$ iff there exists $c \in Z$, such that $S_1=S_2+c$ (see [5]).

Proposition 2.1. *The sets S and S+1 are matching.*

Proof. If $S, S+1$ are not matching there exist $A \subseteq S$, $B \subseteq S+1$ such that $2n \notin I(A)=I(B)$.

If y is the greatest element of $I(A)$ and $\{x,y\} \in A$, it is clear that $y \neq x+1$ and $\{x+1,y+1\} \in (S+1) \setminus B$. Moreover, we have that

$$|I(B) \cap [x+1,y+1]| = |(I(A) \cap [x,y]) \setminus \{x\}| = |I(A) \cap [x,y]| - 1,$$

though every set $I(K)\cap[\alpha,\beta]$ with $\{\alpha,\beta\}\in S$ has even cardinality. Hence we obtain the desired contradiction. ■

The following result shows that the translation of nested sets preserves the matching property. The proof is straightword and it is omitted.

Proposition 2.2. *If S_1,S_2 are matching, then S_1+c, S_2+c are also matching.*

A pair $\{\alpha,b\}\in S$ is called *outer pair* if there is no pair $\{c,d\}\in S$ such that $c<\alpha<b<d$. Obviously, every nested set S contains at least one outer pair.

In [5] it has been shown that every element of N_{2n+2} may be written in either one or the other of the forms :

$$S^*=S\cup\{\{2n+1,2n+2\}\} \text{ and } S_{\alpha b}=(S\setminus\{\{\alpha,b\}\})\cup\{\{\alpha,2n+2\},\{b,2n+1\}\}.$$

where $S\in N_{2n}$ and $\{\alpha,b\}$ is an outer pair of S .

These representations are used in the following results for the construction of matching nested sets.

Proposition 2.3. *If the sets R, S of N_{2n} are matching and $\{\alpha,b\}$ is an outer pair, then the sets R^*, $S_{\alpha b}$ of N_{2n+2} are also matching.*

Proof. It is enough to show that, given $A\subseteq R^*$ and $B\subseteq S_{\alpha b}$ with $2n+2\in I(A)=I(B)$, then $I(A)=[2n+2]$.

We first note that since $\{2n+1,2n+2\}\in R^*$ and $\{\alpha,2n+2\}\in S_{\alpha b}$, we have that $2n+1$, $\alpha\in I(A)$. Similarly, since $\{b,2n+1\}\in S_{\alpha b}$, we deduce that $b\in I(A)$.

We now consider the following sets :

$$A'=A\setminus\{\{2n+1,2n+2\}\}\subseteq R,$$

$$B'=(B\setminus\{\{\alpha,2n+2\},\{b,2n+1\}\})\cup\{\{\alpha,b\}\}\subseteq S.$$

It follows that $I(A')=I(A)\setminus\{2n+1,2n+2\}= I(B)\setminus\{2n+1,2n+2\}=I(B')$.

Then since R,S are matching and $I(A')\neq\emptyset$ we obtain that $I(A')=[2n]$; hence $I(A)=[2n+2]$, so that R^*, $S_{\alpha b}$ are matching. ■

Proposition 2.4. *If for the sets R,S of N_{2n} there exist partitions R_1,R_2 and S_1,S_2 respectively, such that the pairs R_1,S_1 and R_2,S_2 are matching, then the sets R_{ab} and S_{cd} of N_{2n+2} with $\{a,b\}\in R_1$ and $\{c,d\}\in S_2$ are also matching.*

Proof. It is enough to show that, given $A \subseteq R_{ab}$ and $B \subseteq S_{cd}$ with $2n+2 \in I(A)=I(B)$, then $I(A)=[2n+2]$.

We first show that $2n+1 \in I(A)$. Since $S_1 \subset S_{cd}$ and R_1, S_1 are matching, we have : $I(S_1 \cap B)=I(S_1) \cap I(B)=I(R_1) \cap I(A) \supseteq I(R_1 \cap A)$.

Moreover, since the cardinalities of the sets $I(S_1 \cap B)$ and $I(R_1 \cap A)$ are even and the only elements of $I(R_1) \cap I(A)$ that might not belong to $I(R_1 \cap A)$ are α and b, we obtain that :

$$\alpha \in (I(R_1) \cap I(A)) \setminus I(R_1 \cap A) \text{ iff } b \in (I(R_1) \cap I(A)) \setminus I(R_1 \cap A)$$

Thus, since α clearly belongs to the above difference, so does b. This shows that $b \in I(A)$ and hence $2n+1 \in I(A)$.

We now consider the following sets :

$$A'=(A \setminus \{\{\alpha,2n+2\},\{b,2n+1\}\}) \cup \{\{\alpha,b\}\} \subseteq R,$$
$$B'=(B \setminus \{\{c,2n+2\},\{d,2n+1\}\}) \cup \{\{c,d\}\} \subseteq S.$$

We have that $I(A')=I(A) \setminus \{2n+1,2n+2\}=I(B) \setminus \{2n+1,2n+2\}=I(B')$.

Moreover, since $I(A' \cap R_1)=I(A') \cap I(R_1)=I(B') \cap I(S_1)=I(B' \cap S_1) \neq \emptyset$ and R,S are matching, we obtain that $A' \cap R_1=R_1$ and $B' \cap S_1=S_1$.

Similarly it is shown that $A' \cap R_2=R_2$ and $B' \cap S_2=S_2$, so that $A'=R$ and $B'=S$. This shows that $A=R_{\alpha b}$, $B=S_{cd}$ and $I(A)=I(B)=[2n+2]$. ∎

The above two propositions suggest two constructions for the generation of matching nested sets, using matching nested sets of smaller size. In the following result it is shown that the converse procedure is also valid.

Proposition 2.5. *Every pair of matching nested sets of N_{2n+2} may be generated by matching nested sets of smaller size.*

Proof. If U,L is a pair of matching nested sets of N_{2n+2}, we consider the following two cases :

α) Either $\{2n+1,2n+2\} \in U$ or $\{2n+1,2n+2\} \in L$.

Without loss of generality we assume that $\{2n+1,2n+2\} \in U$; let $\alpha, b \in [2n]$ such that $\{\alpha,2n+2\} \in L$ and $\{b,2n+1\} \in L$.

If $R=U \setminus \{\{2n+1,2n+2\}\}$ and $S=(L \setminus \{\{\alpha,2n+2\},\{b,2n+1\}\}) \cup \{\{\alpha,b\}\}$, it follows that R,S are matching nested sets of N_{2n}, $\{\alpha,b\}$ is an outer pair of S and $R^*=U$, $S_{\alpha b}=L$.

β) $\{2n+1,2n+2\} \notin U$ and $\{2n+1,2n+2\} \notin L$.

Then, there exist $\alpha,b,c,d \in [2n]$ such that :

$$\{\alpha,2n+2\} \in U,\ \{b,2n+1\} \in U,\ \{c,2n+2\} \in L,\ \{d,2n+1\} \in L.$$

We construct two finite sequences $(x_k),(y_k)$ in $[2n]$ as follows :

Let (x_k), $k\in[v]$ be the sequence in $[2n]$, with maximum length, which satisfies the properties :

$$x_1=c, \quad \{x_k,x_{k+1}\}\in U \text{ if } k \text{ is odd}, \quad \{x_k,x_{k+1}\}\in L \text{ if } k \text{ is even}.$$

We will show that $x_v=d$. Firstly, we prove that $\{x_{v-1},x_v\}\in U$; indeed, otherwise $\{x_{v-1},x_v\}\in L$ and if $\xi\in[2n+2]$ with $\{x_v,\xi\}\in U$, by the maximality of v we deduce that $\xi\notin[2n]$ so that $\xi=2n+2$ and $x_v=c$. Thus, if $A=\{\{x_k,x_{k+1}\} : k \text{ odd}\}$ and $B=\{\{x_k,x_{k+1}\} : k \text{ even}\}$, it follows that $A\subseteq U$, $B\subseteq L$ and $\emptyset\neq I(A)=I(B)\neq[2n+2]$, which contradicts the matching property of U,L.

Now, if $w\in[2n+2]$ with $\{x_v,w\}\in L$, by the maximality of v we must have that $w\notin[2n]$, so that $w=2n+1$ and $x_v=d$.

Similarly, we define (y_k), $k\in[t]$ such that $y_1=\alpha$, $y_t=b$, $\{y_k,y_{k+1}\}\in L$ if k is odd, $\{y_k, y_{k+1}\}\in U$ if k is even.

Let
$$R=(U\setminus\{\{\alpha,2n+2\},\{b,2n+1\}\})\cup\{\{\alpha,b\}\},$$
$$S=(L\setminus\{\{c,2n+2\},\{d,2n+1\}\})\cup\{\{c,d\}\},$$
$$R_1=\{\{x_i,x_{i+1}\} : i \text{ is odd}, i\leq v-1\}\subseteq R,$$
$$R_2=\{\{y_i,y_{i+1}\} : i \text{ is even}, i<t-1\}\cup\{\{\alpha,b\}\}\subseteq R,$$
$$S_1=\{\{x_i,x_{i+1}\} : i \text{ is even}, i<v-1\}\cup\{\{c,d\}\}\subseteq S,$$
$$S_2=\{\{y_i,y_{i+1}\} : i \text{ is odd}, i\leq t-1\}\subseteq S.$$

It is easy to check that the sets R_1,R_2 and S_1,S_2 form partitions of R,S respectively, the pairs R_1,S_1 and R_2,S_2 are matching and $U=R_{\alpha b}$, $L=S_{cd}$. ∎

3. Planar permutations

A permutation σ on $[2n]$ with $\sigma(1)=1$ is called *planar* (p.p.) if the sets, $U_\sigma=\{\{\sigma(2i-1),\sigma(2i)\}: i\in[n]\}$, $L_\sigma=\{\{\sigma(2i),\sigma(2i+1)\}: i\in[n-1]\}\cup\{\{\sigma(2n),1\}\}$ are both nested.

Let Π_{2n} be the set of all p.p. of size 2n. It is easy to check that for every $\sigma\in\Pi_{2n}$ we have $\sigma(i)$ is odd iff i is odd.

The *obverse* σ^*, the *conjugate* σ^c and the *additive shift* σ^s of $\sigma\in\Pi_{2n}$ are the permutations on $[2n]$ with $\sigma^*(i)=\sigma(2n+2-i)$, $\sigma^c(i)=2n+2-\sigma(i)$ and $\sigma^s(i)=1+\sigma(r)$, where $r=\sigma^{-1}(2n)+i-1$, $i\in[2n]$.

The permutation σ^s may be obtained from σ in a practical way, if we increase the elements of σ by one and then we attach the part found on the left of 1 to the right part. For example from the permutation σ=16783452 we take 278|14563 and σ^s=14563278.

Clearly, since

$U_{\sigma^*} = L_\sigma$, $L_{\sigma^*} = U_\sigma$, $U_{\sigma^c} = -U_\sigma+2n+2$, $L_{\sigma^c} = -L_\sigma+2n+2$, $U_{\sigma^s} = L_\sigma+1$, $L_{\sigma^s} = U_\sigma+1$, we have the following result.

Proposition 3.1. *The obverse, the conjugate and the additive shift of any p.p. is a p.p.*

Since the conjugate of a p.p. is a p.p., the determination of the elements of Π_{2n} is reduced to the half, i.e., to the set of planar permutations with $\sigma(2)\leq n$ (resp. $\sigma(2)\leq n+1$) when n is even (resp. odd).

We now consider iterations of the additive shifting operation :

$\sigma^1=\sigma^s$, $\sigma^k=(\sigma^{k-1})^s$, $k\geq 2$; (we assume also that $\sigma^0=\sigma$).

Iteration satisfies the following properties :

(i) $\sigma^{2n}=\sigma$

(ii) $\sigma^{k+\lambda}=(\sigma^k)^\lambda=(\sigma^\lambda)^k$

(iii) If $\sigma^p=\sigma$ then $\sigma^{\alpha p}=\sigma$, $\forall\ \alpha\in N^*$

Furthermore, by induction we can easily check that if $\sigma\in\Pi_{2n}$, then :

$$U_{\sigma^k}=\begin{cases} U_\sigma+k & \text{if k is even} \\ L_\sigma+k & \text{if k is odd}\end{cases} \quad \text{and} \quad L_{\sigma^k}=\begin{cases} L_\sigma+k & \text{if k is even} \\ U_\sigma+k & \text{if k is odd}\end{cases}$$

Further, an equivalence relation on Π_{2n} is defined by $\sigma\approx\tau$ iff there exists $k\in N$ such that $\sigma^k=\tau$.

We say that a p.p. $\sigma\in\Pi_{2n}$ has *shifting order* p iff p is the least positive integer such that $\sigma^p=\sigma$. It is evident that σ has shifting order p iff the equivalence class of σ is the set $\{\sigma^k : k=0,1,\ldots,p-1\}$.

Following a similar argument to the proof of proposition 2.3 of [5], we obtain the following result.

Proposition 3.2. *If $\sigma \in \Pi_{2n}$ has shifting order p, then $\sigma^m = \sigma$ iff p divides m.*

We note that in particular p divides 2n. Furthermore, p=1 iff $\sigma = \sigma_1$ or $\sigma = \sigma_\omega$, where $\sigma_1(i) = i$ and $\sigma_\omega(i) = 2n+2-i$ for every $i \in [2n]$.

From the above remarks and proposition 3.2 we obtain the following result.

Proposition 3.3. *The set Π_{2n} is partitioned into equivalence classes such that the cardinality of each one is a divisor of 2n and exactly two of them contain only one element.*

We conclude this section by giving the relation between the planar permutations and the matching nested sets.

Proposition 3.4. *U_σ , L_σ are matching for every $\sigma \in \Pi_{2n}$.*

Proof. If $I(A) = I(B) \neq \emptyset$ for $A \subseteq U_\sigma$ and $B \subseteq L_\sigma$ we have $\sigma(i) \in I(A)$ iff $\sigma(i-1) \in I(A)$, for every $i > 1$. This shows that $I(A) = [2n]$, so that U_σ, L_σ are matching. ∎

We now proceed to the converse.

Proposition 3.5. *Given two matching nested sets $U, L \in N_{2n}$, there exists a unique $\sigma \in \Pi_{2n}$ such that $U = U_\sigma$ and $L = L_\sigma$.*

Proof. We first construct a mapping σ on $[2n+1]$ as follows :

$\sigma(1) = 1$;

$\sigma(2i)$ is the unique element of $[2n]$ such that $\{\sigma(2i-1), \sigma(2i)\} \in U$.

$\sigma(2i+1)$ is the unique element of $[2n]$ such that $\{\sigma(2i+1), \sigma(2i)\} \in L$.

We will show that the restriction of σ on $[2n]$ is the required p.p.

In order to show that σ is a permutation, suppose that there exist $k, \lambda \in [2n]$ such that $k \neq \lambda$ and $\sigma(k) = \sigma(\lambda)$. Clearly, since σ maps every odd (resp. even) number to an odd (resp. even) number, we have that both k, λ are either odd or even. Without loss of generality we assume that k, λ are odd and choose them so that $k < \lambda$ and $\lambda - k$ is minimum.

Then, we can easily check that $\sigma(s) \neq \sigma(t)$ for every s,t with $k < s < t < \lambda$.

Thus, the sets

$A=\{\{\sigma(s),\sigma(s+1)\} : k\leq s<\lambda-1, \text{ s odd}\}$,

$B=\{\{\sigma(s),\sigma(s+1)\} : k<s\leq\lambda-1, \text{ s even}\}$,

are well defined, non-empty subsets of U,L respectively.

Furthermore, $I(A)=I(B)$, $|I(A)|=(\lambda-1)-k+1=\lambda-k$ and $0<|I(A)|<2n$. Thus, $I(A)\neq\emptyset$ and $I(A)\neq[2n]$, contradicting the matching property of U,L; so σ is 1-1.

Now since, by the construction of σ, $U_\sigma\subseteq U$ and $|U|=n=|U_\sigma|$ we obtain $U=U_\sigma$.

The proof of $L=L_\sigma$ is similar provided that $\sigma(2n+1)=1$. For the proof of the last equality, let i be the unique element of [n] such that $\sigma(2n+1)=\sigma(2i-1)$. Then, for the non-empty sets

$A'=\{\{\sigma(s),\sigma(s+1)\}: 2i-1\leq s\leq 2n-1, \text{ s odd}\}$, $B'=\{\{\sigma(s),\sigma(s+1)\} : 2i\leq s\leq 2n, \text{ s even}\}$,

we have $A'\subseteq U$, $B'\subseteq L$ and $I(A')=I(B')$.

Thus, by the matching property of U,L we obtain that $I(A')=[2n]$, so that $2n-(2i-1)+1=2n$ and hence $i=1$. So, $\sigma(2n+1)=\sigma(2i-1)=\sigma(1)=1$.

This proves the existence of the p.p. σ, and since the uniqueness is obvious, the proof of the proposition is complete. ∎

We note that propositions 3.4, 3.5 and the inductive construction of N_{2n} given in [5], enable us to create an algorithm for the construction of the set Π_{2n}.

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