# Enumeration of certain binary vectors

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**Abstract**

In 1986, W. S. Griffith [2] proposed a reliability model and described a Markov chain approach for computing the reliability in the system. F. K. Hwang and S. Papastavridis [4] gave a closed-form formula for this reliability in 1991, by counting the number of $n-$dimensional binary vectors containing exactly $k$ non-overlapping $m$-tuples of consecutive ones. In 1988, T. M. Apostol [1] established recursive formulas and a generating function for a related problem, the enumeration of $n-$dimensional binary vectors containing exactly $k$ isolated $m$-tuples of consecutive ones. In this paper we study two interrelated elementary enumerator functions in terms of which we are able to address the two enumeration problems mentioned above. This enables us to provide a unified approach to deriving closed-form formulas for both problems, and provides solutions to finer enumeration problems. We extend our results to the case of cyclic binary vectors, and briefly discuss the number of cyclic ordered partitions of a positive integer $w$.

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# 1 Introduction

In 1986, W. S. Griffith [2] proposed a reliability model in which a system consists of $n$ independent and identically distributed components arranged in a line. Each component has probability $p$ of *being operational* and probability $q = 1 - p$ of *having failed.* The system fails if and only if at least $k$ nonoverlapping sets of $m$ consecutive components all fail. In his paper Griffith describes a probabilistic approach, based on Markov chains, for computing the reliability $R_k^{(m)}(n)$ of such a system.

In a binary vector, an $m$-tuple of consecutive ones is said to be *isolated* if no other ones are adjacent to the $m$-tuple. In 1988, T.M. Apostol [1] studied the number $A_k^{(m)}(n)$ of $n$-dimensional binary vectors containing exactly $k$ isolated $m$-tuples of consecutive ones. In his paper Apostol establishes recursive formulas and a generating function for $A_k^{(m)}(n)$, moreover, he tabulates these numbers for $n \leq 25$, $0 \leq k \leq 5$, and $m \leq 5$. Clearly, a closed-form formula for $A_k^{(m)}(n)$ is desirable.

In 1991, F.K. Hwang and S. Papastavridis [4] studied a similar enumeration problem to Apostol's. Their problem arises directly from Apostol's by replacing the term *isolated* by *non-overlapping*, where two $m$-tuples of consecutive ones in a binary vector are said to be *non-overlapping* if they are disjoint. The authors give a closed-form formula for the number $B_k^{(m)}(n)$ of $n$-dimensional binary vectors containing exactly $k$ non-overlapping $m$-tuples of consecutive ones, and also study the cyclic case. Their result directly provides a closed-form formula to Griffith's reliability $R_k^{(m)}(n)$.

In Section 2 of this paper we study two related enumerator functions $f$ and $f_1$, defined below, which are more elementary than $A_k^{(m)}(n)$ and $B_k^{(m)}(n)$. We obtain closed-form formulas for $f$ and $f_1$, and since $A_k^{(m)}(n)$ and $B_k^{(m)}(n)$ can be expressed in terms of $f$ and $f_1$, we obtain closed-form formulas for $A_k^{(m)}(n)$ and $B_k^{(m)}(n)$. Thus, we are able to obtain a closed-form formula for Apostol's problem, and although Hwang and Papastavridis have provided an elegant closed-form formula for $B_k^{(m)}(n)$, our method provides a unified approach to obtaining such closed-form formulas, and solutions to additional finer enumeration questions.

We begin with some terminology and establish our notation. For a given $n$-dimensional binary vector

$$v = (a_1, a_2, \cdots, a_n)$$

a *1-component* (resp. *0-component*) is an element $a_i$ which has the value 1 (resp. 0). The number of 1-components of $v$ is called its *weight*, denoted by $w(v)$, hence

$$w(v) = \sum_{i=1}^{n} a_i.$$

A *1-block* (resp. *0-block*) of $v$ is a segment of consecutive 1-components (resp 0-components) of maximal length. Thus, a 1-block, or a 0-block, can not be extended in $v$ into a longer segment of consecutive 1's (resp. 0's). Let $f(n, m, k, w, b)$ (resp. $f_1(n, m, k, w, b)$) be the number of $n$-dimensional binary vectors of weight

$w$, with $b$ 1-blocks, and $k$ isolated (resp. non-overlapping) $m-$tuples of consecutive 1-components, Further, let $g(n, m, k, w)$ (resp. $g_1(n, m, k, w)$) be the number of $n-$dimensional binary vectors of weight $w$ with $k$ isolated (resp. non-overlapping) $m-$tuples of consecutive 1-components, and $h(n, m, k, b)$ (resp. $h_1(n, m, k, b)$) be the number of $n-$dimensional binary vectors with $b$ 1-blocks and $k$ isolated (resp. non-overlapping) $m-$tuples of consecutive 1-components. Clearly,

$$A_k^{(m)}(n) = \sum_{w=0}^{n} g(n, m, k, w) \tag{1.1}$$

$$A_k^{(m)}(n) = \sum_{b=0}^{\lceil \frac{n}{2} \rceil} h(n, m, k, b) \tag{1.2}$$

$$g(n, m, k, w) = \sum_{b=0}^{\lceil \frac{n}{2} \rceil} f(n, m, k, w, b) \tag{1.3}$$

$$h(n, m, k, b) = \sum_{w=0}^{n} f(n, m, k, w, b) \tag{1.4}$$

$$B_k^{(m)}(n) = \sum_{w=0}^{n} g_1(n, m, k, w) \tag{1.5}$$

$$B_k^{(m)}(n) = \sum_{b=0}^{\lceil \frac{n}{2} \rceil} h_1(n, m, k, b) \tag{1.6}$$

$$g_1(n, m, k, w) = \sum_{b=0}^{\lceil \frac{n}{2} \rceil} f_1(n, m, k, w, b) \tag{1.7}$$

and

$$h_1(n, m, k, w) = \sum_{w=0}^{n} f_1(n, m, k, w, b). \tag{1.8}$$

Therefore, formulas for $f(n, m, k, w, b)$ and $f_1(n, m, k, w, b)$ will provide formulas for all $g(n, m, k, w)$, $g_1(n, m, k, w)$, $h(n, m, k, b)$, $h_1(n, m, k, b)$), $A_k^{(m)}(n)$ and $B_k^{(m)}(n)$. Consequently, finding formulas for $f(n, m, k, w, b)$ and $f_1(n, m, k, w, b)$ is a more fundamental problem than the problems solved respectively by Apostol and by F.K. Hwang and S. Papastavridis.

In Section 3 we discuss the cyclic version of these problems. In Section 4, we apply a lemma established in Section 3 to develop the cyclic counterpart of the well-known formula for the number of integer solutions of the equation

$$x_1 + x_2 + \ldots + x_b = w \tag{1.9}$$

$$x_i \geq 1 \quad (1 \leq i \leq b). \tag{1.10}$$

Throughout this paper the combinatorial meaning of $\binom{u}{v}$ is defined only for integers $u$ and $v$, and $\binom{u}{v} = 0$ if $u < 0$ or $v < 0$ or $u < v$, with the exception that we take $\binom{-1}{-1} = 1$.

## 2 Formulas for $f(m,n,k,w,b)$ and $f_1(n,m,k,w,b)$

We first prove the following result:

**Theorem 2.1** *For any given positive integers* $n$, $m$, $k$, $w$, $b$, *we have*

$$f(n,m,k,w,b) = \binom{n-w+1}{b} \sum_{r \geq k} (-1)^{r-k} \binom{r}{k} \binom{b}{r} \binom{w-rm-1}{b-r-1}. \tag{2.1}$$

Proof: In general, an $n-$dimensional binary vector counted in the number $f(n,m,k,w,b)$ has the pattern $0^{y_1}1^{x_1}0^{y_2}1^{x_2}\cdots 0^{y_b}1^{x_b}0^{y_{b+1}}$, that is:

$$(\underbrace{0,\ldots,0}_{y_1}, \underbrace{1,\ldots,1}_{x_1}, \underbrace{0,\ldots,0}_{y_2}, \underbrace{1,\ldots,1}_{x_2} \cdots, \underbrace{0,\ldots,0}_{y_b}, \underbrace{1,\ldots,1}_{x_b}, \underbrace{0,\ldots,0}_{y_{b+1}}) \tag{2.2}$$

with

$$y_1 + y_2 + \cdots + y_b + y_{b+1} = n - w \tag{2.3}$$

$$y_1 \geq 0, \quad y_i \geq 1 \quad (2 \leq i \leq b), \quad y_{b+1} \geq 0 \tag{2.4}$$

and

$$x_1 + x_2 + \cdots + x_b = w \tag{2.5}$$

$$x_i \geq 1, \quad (1 \leq i \leq b) \tag{2.6}$$

$$\text{where exactly } k \text{ of the } x_i \text{ are equal to } m. \tag{2.7}$$

Clearly, $f(n,m,k,w,b)$ is nothing but the number of patterns (2.2) subject to (2.3)-(2.7). Let $y_1' = y_1 + 1$, $y_i' = y_i$ $(2 \leq i \leq b)$, and $y_{b+1}' = y_{b+1} + 1$. It follows that the number of integer solutions to (2.3) subject to (2.4) is exactly the same as the number of integer solutions of

$$y_1' + y_2' + \cdots + y_b' + y_{b+1}' = n - w + 2 \tag{2.8}$$

subject to

$$y_i' \geq 1 \quad (1 \leq i \leq b+1). \tag{2.9}$$

In turn, the number of integer solutions to (2.8) subject to (2.9) is well known to be

$$\binom{n-w+1}{b} \tag{2.10}$$

(for example, see [3], page 31.) When $r < b$, for any given $r-$set $\{p_1, p_2, \ldots, p_r\}$ of indices such that $1 \leq p_1 < p_2 < \ldots < p_r \leq b$, the number of solutions of (2.5) subject to (2.6) and subject to

$$x_{p_1} = x_{p_2} = \ldots = x_{p_r} = m$$

is the number of solutions of

$$x_{q_1} + x_{q_2} + \ldots + x_{q_{b-r}} = w - rm \tag{2.11}$$
$$\text{with} \quad x_{q_j} \geq 1, \qquad (1 \leq j \leq b - r)$$

where,

$$\{q_1, q_2, \ldots, q_{b-r}\} \cup \{p_1, p_2, \ldots, p_r\} = \{1, 2, \ldots, b\}. \tag{2.12}$$

Hence, by the same principle as for (2.10) this number of solutions is

$$\binom{w - rm - 1}{b - r - 1} \tag{2.13}$$

which is dependent on $r$ but independent of the choice of $p_1, p_2, \ldots, p_r$. When $r = b$ and $w = rm$, there is a single solution and the above formula yields

$$\binom{w - rm - 1}{b - r - 1} = \binom{-1}{-1} = 1.$$

When $r = b$ and $w > rm$ then there are no solutions to (2.11) and formula (2.13) is still correct. By the inclusion-exclusion principle, the number of solutions of (2.5) subject to (2.6) and (2.7) is

$$\sum_{r \geq k} (-1)^{r-k} \binom{r}{k} \binom{b}{r} \binom{w - rm - 1}{b - r - 1}.$$

Therefore, the number of vectors of (2.2) subject to (2.3) – (2.7) is

$$\binom{n - w + 1}{b} \sum_{r \geq k} (-1)^{r-k} \binom{r}{k} \binom{b}{r} \binom{w - rm - 1}{b - r - 1}$$

which is $f(n, m, k, w, b)$. □

We will need the following:

**Lemma 2.1** *Let $a$, $b$, $c$, $p$, $q \in \mathbb{Z}$, such that $a$, $b$, $c$, $p + q > 0$, Then, the number of integral solutions $(u_1, \ldots, u_p, v_1, \ldots, v_q)$ to equation*

$$u_1 + \cdots + u_p + v_1 + \cdots + v_q = c \tag{2.14}$$

*subject to the conditions*

$$1 \leq u_i \leq a, \ \ (1 \leq i \leq p)\ ; \qquad 1 \leq v_j \leq b, \ \ (1 \leq j \leq q) \tag{2.15}$$

*is*

$$N(a, b, c, p, q) = \sum_{s,t \geq 0} (-1)^{s+t} \binom{p}{s} \binom{q}{t} \binom{c - sa - tb - 1}{p + q - 1}. \tag{2.16}$$

Proof: For a non-negative integral vector $x = (u_1, \ldots, u_p, v_1, \ldots, v_q)$, we say that $x$ satisfies property $P_i$ if $u_i > a$, $1 \leq i \leq p$. Similarly $x$ is said to satisfy property $Q_j$ if $v_j > b$. To evaluate the number of solutions to equation (2.14), subject to (2.15) we need to determine the number of solutions to (2.14), satisfying none of the properties $P_1, \ldots, P_p$, $Q_1, \ldots, Q_q$. Let $N(P_{i_1}, \ldots, P_{i_s};\ Q_{j_1}, \ldots, Q_{j_t})$ be the number of solutions $x$ to (2.14) satisfying properties $P_{i_1}, \ldots, P_{i_s}, Q_{j_1}, \ldots, Q_{j_t}$, and let

$$N_{s,t} = \sum_{\substack{1 \le i_1 \le \ldots \le i_s \le p \\ 1 \le j_1 \le \ldots \le j_t \le q}} N(P_{i_1}, \ldots, P_{i_s}; Q_{j_1}, \ldots, Q_{j_t}). \tag{2.17}$$

Then, it is clear that

$$N_{s,t} = \binom{p}{s}\binom{q}{t}\binom{c - sa - tb - 1}{p + q - 1}. \tag{2.18}$$

Hence, by the inclusion-exclusion principle the lemma follows. □

**Theorem 2.2** *For arbitrary positive integers $n$, $m$, $k$, $w$ and $b$,*

$$f_1(n, m, k, w, b) = \binom{n - w + 1}{b} \sum_{r \ge 0} \binom{b}{r}\binom{k - 1}{b - r - 1} N(m-1, m, w-km+b-r, r, b-r). \tag{2.19}$$

Proof: An $n-$dimensional binary vector counted in $f_1(n, m, k, w, b)$ has pattern (2.2) subject to (2.3), (2.4) and

$$t_i + z_i \ge 1 \quad (1 \le i \le b) \tag{2.20}$$

$$t_1 + t_2 + \ldots + \ldots t_b = k \tag{2.21}$$

where,

$$x_i = t_i m + z_i, \quad t_i = \lfloor \frac{x_i}{m} \rfloor. \tag{2.22}$$

This is because there are exactly $\lfloor \frac{x}{m} \rfloor$ non-overlapping $m-$tuples of consecutive ones in a 1-block of length $x$. Suppose there are exactly $r$ of the $t_i$'s that are zero, say $t_{p_1} = t_{p_2} = \ldots = t_{p_r} = 0$. Then, (2.21) becomes

$$t_{q_1} + t_{q_2} + \ldots + t_{q_{b-r}} = k \tag{2.23}$$

$$t_{q_i} \ge 1 \quad (1 \le i \le b - r) \tag{2.24}$$

and,

$$z_{p_1} + \ldots + z_{p_r} + z_{q_1} + \ldots + z_{q_{b-r}} = w - (t_{q_1} + \ldots + t_{q_{b-r}})m = w - km \tag{2.25}$$

$$m - 1 \ge z_{p_i} \ge 1, \ (1 \le i \le r), \qquad m - 1 \ge z_{q_j} \ge 0, \tag{2.26}$$

where $p_1, \ldots, p_r$, $q_1, \ldots, q_{b-r}$ satisfy (2.12). For a given choice of indices $1 \le p_1 < p_2 < \ldots < p_r \le b$, the number of solutions of (2.23) subject to (2.24) is

$$\binom{k - 1}{b - r - 1}. \tag{2.27}$$

By defining new variables $z'_{q_j} = 1 + z_{q_j}$, for $1 \le j \le b - r$, we obtain a system of equations equivalent to (2.25), (2.26) as follows:

$$z_{p_1} + \ldots + z_{p_r} + z'_{q_1} + \ldots + z'_{q_{b-r}} = w - km + b - r \tag{2.28}$$

$$m - 1 \ge z_{p_i} \ge 1, \quad (1 \le i \le r), \qquad m \ge z'_{q_j} \ge 1, \quad (1 \le j \le b - r). \tag{2.29}$$

Hence, the number of solutions of (2.25) subject to (2.26) is equal to:

$$N(m-1, m, w - km + b - r, r, b - r). \tag{2.30}$$

It follows that for any particular choice of $p_1, p_2, \ldots, p_r$, the number of solutions of (2.20)-(2.22) is

$$\binom{k-1}{b-r-1} N(m-1, m, w - km + b - r, r, b - r). \tag{2.31}$$

Since there are $\binom{b}{r}$ ways of choosing $p_1, p_2, \ldots, p_r$, we obtain (2.19) by (2.10) and (2.31). $\square$

We illustrate the statement of Theorem (2.2) by some examples for small values of $k$ and $b$.

**Example 2.1** *When $k = b = 1$, $n \ge w \ge 1$, $m \ge 1$ the expression for $f_1(n, m, 1, w, 1)$ in Theorem (2.2) reduces to*

$$(n - w + 1)\binom{1}{0}\binom{0}{0}\sum_{s,t \ge 0}(-1)^{s+t}\binom{0}{s}\binom{1}{t}\binom{w - m - tm}{0}$$

$$= (n-w+1)\sum_{t \ge 0}(-1)^t\binom{1}{t}\binom{w - m - tm}{0} = (n-w+1)(\binom{w-m}{0} - \binom{w-m-m}{0}).$$

This evaluates to 0 if $w < m$ or $w \ge 2m$, and $n - w + 1$ if $m \le w < 2m$. This is the correct count as can be easily verified from the figure below.

$$(\underbrace{0, \ldots, 0,}_{y_1}\ \underbrace{1, \ldots, 1,}_{w}\ \underbrace{0, \ldots, 0}_{y_2})$$

**Example 2.2** *When $k = 3$, $b = 2$, $n \ge w \ge 1$, $m \ge 1$ the expression for $f_1(n, m, 1, w, 1)$ in Theorem (2.2) reduces to*

$$(n - w + 1)\binom{1}{0}\binom{0}{0}\sum_{s,t \ge 0}(-1)^{s+t}\binom{0}{s}\binom{1}{t}\binom{w - m - tm}{0}$$

$$= (n-w+1)\sum_{t \ge 0}(-1)^t\binom{1}{t}\binom{w - m - tm}{0} = (n-w+1)(\binom{w-m}{0} - \binom{w-m-m}{0}).$$

As analyzed in Section 1, all functions displayed in (1.1) - (1.8) now have closed-form formulas. We just exhibit the one for (1.1) since it was asked for in the literature.

**Theorem 2.3** *If $n$, $m$, $k$, $w$, and $b$ are arbitrary positive integers, then:*

$$A_k^{(m)} = \sum_{w=0}^{n} \sum_{b=0}^{\lceil \frac{n}{2} \rceil} \binom{n-w+1}{b} \sum_{r \geq k} (-1)^{r-k} \binom{r}{k} \binom{b}{r} \binom{w-rm-1}{b-r-1}.$$

# 3 The cyclic case

In the cyclic case, we are dealing with a binary vector $v = (a_1, a_2, \dots, a_n)$, where $a_n$ is considered to be adjacent to $a_1$. It is clear that two elements which are distinct when viewed as linear vectors, may be equivalent under a cyclic shift. For example the two vectors $(0,0,0,1,1,1,1,0,0,0,1,1,1,1)$ and $(1,1,1,1,0,0,0,1,1,1,1,0,0,0)$ are equivalent.

Let $\psi(n,m,k,w,b)$ (resp $\psi_1(n,m,k,w,b)$) be functions defined just as $f$ and $f_1$ but now for the cyclic case. For example, $\psi(n,m,k,w,b)$ counts the number of equivalent classes under cyclic shifts of $n-$dimensional binary vectors of weight $w$ with $b$ 1-blocks and $k$ isolated $m-$tuples of 1's. Similarly, let $\gamma(n,m,k,w)$ and $\gamma_1(n,m,k,w)$) be the cyclic analogues of $g(n,m,k,w)$ and $g_1(n,m,k,w)$ respectively, and $\eta(n,m,k,b)$ and $\eta_1(n,m,k,b)$ the cyclic analogues of functions $h(n,m,k,b)$ and $h_1(n,m,k,b)$. We begin by considering $\psi(n,m,k,w,b)$.

Since we are considering the cyclic case, we may restrict our attention to vectors of the form $1^{x_1}0^{y_1}1^{x_2}0^{y_2}\dots 1^{x_b}0^{y_b}$ where $\sum_{i=1}^{b}(x_i+y_i) = n$, $x_i \geq 1$, $y_i \geq 1$. These are equivalent to vectors in (2.2) where $y_{b+1} = 0$. It is convenient to denote vector $1^{x_1}0^{y_1}1^{x_2}0^{y_2}\dots 1^{x_b}0^{y_b}$ simply by

$$(x_1, y_1, x_2, y_2, \cdots, x_b, y_b) = [x, y] \tag{3.1}$$

where $x = (x_1, x_2, \dots, x_b)$, $y = (y_1, y_2, \dots, y_b)$ with the $x_i$ and $y_i$ positive integers. For a given fixed $w \leq n$ let $X$ denote the set of all positive integer vectors $x = (x_1, x_2, \dots, x_b)$ satisfying

$$x_1 + x_2 + \cdots + x_b = w \tag{3.2}$$

and similarly let $Y$ denote the set of all positive integer vectors $y = (y_1, y_2, \dots, y_b)$ satisfying

$$y_1 + y_2 + \cdots + y_b = n - w. \tag{3.3}$$

Then, the collection of all binary vectors of the form $1^{x_1}0^{y_1}\dots 1^{x_b}0^{y_b}$, of weight $w$ with $b$ 1-blocks, can be viewed as the cartesian product $Z = X \times Y$. If $G$ is any permutation group on the symbols $\{1, 2, \dots, b\}$ define a group action $G|X$ of $G$ on $X$ by

$$(x_1, x_2, \dots, x_b)^g = (x_{1^g}, x_{2^g}, \dots, x_{b^g}), \qquad g \in G. \tag{3.4}$$

Similarly, we define a corresponding group action $G|Y$. We proceed to state the following:

**Lemma 3.1** *Let $b$, $w$ be positive integers, and $z$ a divisor of $b$. Let $\pi = (1, 2, \ldots, b)$ and $\sigma$ be any element of order $z$ in the cyclic group generated by $\pi$. If $z$ divides $w$, then, the number of vectors $x = (x_1, x_2, \ldots, x_b)$ fixed by $\sigma$ such that $x_1+x_2+\ldots+x_b = w$ is equal to:*

$$\binom{\frac{w}{z}-1}{\frac{b}{z}-1}. \tag{3.5}$$

*This number is 0 if $z$ does not divide $w$.*

Proof: Let $s = b/z$. Without loss of generality let $\sigma = \pi^s = (1, 1+s, 1+2s, \ldots, 1+(z-1)s)(2, 2+s, 2+2s, \ldots, 2+(z-1)s)\ldots(s, 2s, \ldots, zs)$. Then, $x = (x_1, x_2, \ldots, x_b) \in X$ is fixed by $\sigma$ if and only if $x_i = x_j$ whenever $i$ and $j$ are in the same cycle of $\sigma$. If this happens we say that $x \in X$ is *constant* on the cycles of $\sigma$. It follows that $\sigma$ fixes $(x_1, \ldots, x_b)$ if and only if

$$\begin{array}{ccccccccc}
x_1 & = & x_{1+s} & = & x_{1+2s} & = & \ldots & = & x_{1+(z-1)s} \\
x_2 & = & x_{2+s} & = & x_{2+2s} & = & \ldots & = & x_{2+(z-1)s} \\
. & & . & & . & & \ldots & & . \\
. & & . & & . & & \ldots & & . \\
. & & . & & . & & \ldots & & . \\
x_s & = & x_{2s} & = & x_{3s} & = & \ldots & = & x_{zs}.
\end{array} \tag{3.6}$$

That is, if and only if,

$$zx_1 + zx_2 + \ldots + zx_s = w \tag{3.7}$$

which occurs if and only if $(x_1, x_2, \ldots, x_s)$ is a solution to

$$x_1 + x_2 + \ldots + x_s = w/z. \tag{3.8}$$

If $z$ does not divide $w$, then, trivially, there are no solutions to (3.7), otherwise the number of solutions to (3.7) in $X$ is:

$$\binom{\frac{w}{z}-1}{\frac{b}{z}-1}.$$

Since every element of order $z$ in $G$ has exactly $b/z$ cycles of length $z$ and no other cycles, formula (3.5) holds for any element of order $z$ in $G$. □

Let $G$ be the cyclic group generated by $\pi = (1, 2, \ldots, b)$ as above. $G$ acts on $Z = X \times Y$ by $[x, y]^g = [x^g, y^g]$ for $x \in X$, $y \in Y$, $g \in G$. It is not difficult to see that $\psi(n, m, k, w, b)$ is the number of $G$-orbits on a certain proper subset of $Z$. We are now able to state the following:

**Theorem 3.1** *If $n$, $m$, $k$, $b$, and $w$ are arbitrary positive integers, and $D = gcd\ (n, k, b, w)$, then*

$$\psi(n,m,k,w,b) = \frac{1}{b}\sum_{z|D}\phi(z)\binom{\frac{n-w}{z}-1}{\frac{b}{z}-1}\sum_{r\geq k/z}(-1)^{r-\frac{k}{z}}\binom{r}{\frac{k}{z}}\binom{\frac{b}{z}}{r}\binom{\frac{w}{z}-rm-1}{\frac{b}{z}-r-1} \tag{3.9}$$

*where the outer sum is taken over all divisors $z$ of $D$, and $\phi(z)$ is Euler's totient.*

Proof: Let $\pi = (1, 2, 3, \ldots, b)$. Then, the cyclic group $G$ generated by $\pi$ contains $\phi(z)$ elements of order $z$ for each divisor $z$ of $b$. The cycle decomposition of an element $\sigma$ of order $z$ in $G$ consists of exactly $s = b/z$ cycles of length $z$, and no other cycles. By an application of lemma 3.1 the number of elements $y = (y_1, \ldots, y_b) \in Y$ such that

$$y = (y_1, \ldots, y_b) \text{ is fixed by } \sigma \tag{3.10}$$

and

$$y_1 + y_2 + \ldots + y_b = n - w \tag{3.11}$$

is 0 if $z$ does not divide $n - w$, and

$$\binom{\frac{n-w}{z}-1}{\frac{b}{z}-1} \tag{3.12}$$

otherwise. We now count the number of elements of $X$ such that:

$$x = (x_1, x_2, \ldots, x_b) \text{ is fixed by } \sigma \tag{3.13}$$

and

$$\text{exactly } k \text{ of the } x_i \text{ are equal to } m. \tag{3.14}$$

An element $x = (x_1, x_2, \ldots, x_b)$ is fixed by $\sigma$ if and only if $x$ is constant on the cycles of $\sigma$. Thus, if $z$ does not divide $k$ there are no $x \in X$, subject to (3.14), fixed by $\sigma$. On the other hand, if $z|k$, let $k = tz$, and select an integer $r$

$$k/z = t \leq r \leq s = b/z. \tag{3.15}$$

Suppose $x = (x_1, \ldots, x_b)$ is fixed by $\sigma$ and has some $r$ cycles of $\sigma$ *filled* with $m$'s. Then, on the $rz$ indices $i_1, i_2, \ldots, i_{rz}$ of these cycles

$$x_{i_1} = x_{i_2} = \ldots = x_{i_{rz}} = m. \tag{3.16}$$

Moreover, the complementary indices $\{q_1, \ldots, q_{b-rz}\} = \{1, 2, \ldots, b\} - \{i_1, \ldots, i_{r_z}\}$ fall into $s - r$ cycles of $\sigma$ on which $x$ is constant. Let $q_{j_1}, \ldots, q_{j_{s-r}}$ be a system of representatives of these $s - r$ cycles. Then we have

$$zx_{q_{j_1}} + \ldots + zx_{q_{j_{s-r}}} = w - rmz. \tag{3.17}$$

If $z$ does not divide $w$ then there are no solutions to (3.17). Otherwise, (3.17) becomes

$$x_{q_{j_1}} + \ldots + x_{q_{j_{s-r}}} = \frac{w}{z} - rm. \tag{3.18}$$

In all, (3.18) can be achieved in

$$\binom{b/z}{r}\binom{w/z - rm - 1}{b/z - r - 1} \tag{3.19}$$

distinct ways. Now, by applying the inclusion-exclusion principle, the number of elements $x \in X$ fixed by $\sigma$ with exactly $k$ $x_i$ equal to $m$ is:

$$\sum_{r \geq k/z} (-1)^{r-\frac{k}{z}} \binom{r}{\frac{k}{z}} \binom{b/z}{r} \binom{w/z - rm - 1}{b/z - r - 1}. \tag{3.20}$$

Hence, if $\sigma$ is an element of order $z|D$, it fixes in all

$$\binom{\frac{n-w}{z} - 1}{\frac{b}{z} - 1} \sum_{r \geq k/z} (-1)^{r-\frac{k}{z}} \binom{r}{\frac{k}{z}} \binom{b/z}{r} \binom{w/z - rm - 1}{b/z - r - 1} \tag{3.21}$$

vectors $v = (x_1, y_1, x_2, y_2, \ldots, x_b, y_b)$ of weight $w$, where exactly $k$ of the $x_i$ are equal to $m$. An application of the Cauchy - Frobenius Theorem yields the conclusion of our theorem. □

Next, we turn our attention to the *non-overlapping* cyclic case. Here, we are interested in determining the number of equivalence classes, under cyclic shifts, of binary vectors of dimension $n$, weight $w$, with $b$ 1-blocks and $k$ non-overlapping $m$-tuples of 1-components. As before, these can be viewed as positive integer vectors of the form $v = (x_1, y_1, x_2, y_2, \ldots, x_b, y_b) = [x, y]$, where $x \in X$ and $y \in Y$ subject to the additional conditions (2.13),(2.14), and (2.15), counted up to equivalence under cyclic shifts. Hence, we are interested in the number of $[x, y] \in Z$ fixed by $\sigma \in \langle\pi\rangle$, of order $z$ dividing $b$, where $\pi = (1, 2, \ldots, b)$. We have:

$$x^\sigma = x = (x_1, \ldots, x_b), \quad \text{and} \quad y^\sigma = y = (y_1, \ldots, y_b) \tag{3.22}$$

$$y_1 + y_2 + \ldots + y_b = n - w \tag{3.23}$$

$$x_1 + x_2 + \ldots + x_b = w \tag{3.24}$$

$$x_i = t_i m + z_i, \quad t_i = \lfloor \frac{x_i}{m} \rfloor, \quad 0 \leq z_i \leq m - 1, \quad 1 \leq i \leq b \tag{3.25}$$

$$t_1 + t_2 + \ldots + t_b = k \tag{3.26}$$

$$t_i + z_i \geq 1, \quad 1 \leq i \leq b. \tag{3.27}$$

We state the following:

**Theorem 3.2** *If $n$, $m$, $k$, $b$, and $w$ are arbitrary positive integers, and $D = gcd\,(n, k, b, w)$, then $\psi_1(n, m, k, w, b) =$*

$$\frac{1}{b}\sum_{z|D}\phi(z)\binom{(n-w-z)/z}{(b-z)/z}\sum_{r\geq 0}\binom{\frac{b}{z}}{r}\binom{(k-z)/z}{(b-rz-z)/z}N(m-1, m, s-r+(w-km)/z, r, s-r) \tag{3.28}$$

*where $N(a, b, c, p, q)$ is defined in (2.16), the outer sum is taken over all divisors $z$ of $D$, and $\phi(z)$ is Euler's totient.*

Proof: The number of $y \in Y$, subject to (3.23), and fixed by $\sigma$ has already been calculated and is given by (3.12). Moreover, if $x \in X$ satisfies (3.24)-(3.27), and $x$ is fixed by $\sigma$, then $x$ is constant on the cycles of $\sigma$, so if we revise our notation and denote by $x_i$ the common values of $x$ on the $i^{th}$ cycle of $\sigma$, $x$ can be loosely denoted by $x = x_1^z x_2^z \ldots x_s^z$, with $s = b/z$ and $x_i$ not necessarily distinct from $x_j$ when $i \neq j$. Thus, certain $z$ entries of $x$ are equal to $x_1$, certain $z$ entries equal to $x_2$, etc. We have:

$$zx_1 + zx_2 + \ldots + zx_s = w. \tag{3.29}$$

That is,

$$z(t_1m + z_1) + z(t_2m + z_2) + \ldots + z(t_s + z_s) = w \tag{3.30}$$

which implies:

$$m\sum_{i=1}^{s} t_i + \sum_{i=1}^{s} z_i = w/z \tag{3.31}$$

where,

$$z\sum_{i=1}^{s} t_i = k. \tag{3.32}$$

If $z$ does not divide $k$ then there are no solutions to (3.32). Suppose that $z|k$, then,

$$t_1 + t_2 + \ldots + t_s = k/z. \tag{3.33}$$

For a given $r$, such that $k/z \leq r \leq s = b/z$, suppose that exactly $r$ of the $t_i$ are 0, say $t_{p_1} = t_{p_2} = \ldots = t_{p_r} = 0$. Then, on the complementary set on indices, $\{q_1, \ldots, q_{s-r}\}$ $= \{1, 2, \ldots, s\} - \{p_1, \ldots, p_r\}$ we have:

$$t_{q_1} + t_{q_2} + \ldots + t_{q_{s-r}} = k/z, \quad \text{with} \quad t_{q_i} \geq 1. \tag{3.34}$$

So, (3.31) and (3.34) imply

$$z_{p_1} + \ldots + z_{p_r} + z_{q_1} + \ldots + z_{q_{s-r}} = w/z - m\sum_{j=1}^{s-r} t_{q_j} = \frac{w - km}{z} \tag{3.35}$$

where,

$$m - 1 \geq z_{p_i} \geq 1, \quad 1 \leq i \leq r, \quad \text{and} \quad m - 1 \geq z_{q_j} \geq 0, \quad 1 \leq j \leq s - r. \tag{3.36}$$

Now, (3.34) can be done in

$$\binom{\frac{k}{z}-1}{\frac{b}{z}-r-1} \tag{3.37}$$

ways. Moreover, by defining a new set of variables $z'_{q_j} = 1 + z_{q_j}$ we obtain:

$$z_{p_1} + \ldots z_{p_r} + z'_{q_1} + \ldots + z'_{q_{s-r}} = \frac{w-km}{z} + s - r \tag{3.38}$$

where $m-1 \geq z_{p_i} \geq 1, \quad 1 \leq i \leq r$; the number of solutions to (3.38) is:

$$N(m-1, m, s-r+(w-km)/z, r, s-r). \tag{3.39}$$

Therefore, for a given $r$ and a particular choice of $p_1, p_2, \ldots, p_r$, the number of solutions is:

$$\binom{k/z-1}{b/z-r-1} N(m-1, m, s-r+(w-km)/z, r, s-r). \tag{3.40}$$

Hence, over all possible choices for $p_1, \ldots, p_r$ the number of solutions is

$$\binom{s}{r}\binom{k/z-1}{b/z-r-1} N(m-1, m, s-r+(w-km)/z, r, s-r). \tag{3.41}$$

Consequently, the number of $x \in X$ fixed by $\sigma$, subject to (3.24) - (3.27) is:

$$\sum_{r\geq 0}\binom{\frac{b}{z}}{r}\binom{(k-z)/z}{(b-rz-z)/z} N(m-1, m, s-r+(w-km)/z, r, s-r). \tag{3.42}$$

Hence, putting together (3.12) and (3.42), we obtain the number of $[x, y] \in Z$ fixed by $\sigma$ as:

$$\binom{(n-w-z)/z}{(b-z)/z}\sum_{r\geq 0}\binom{\frac{b}{z}}{r}\binom{(k-z)/z}{(b-rz-z)/z} N(m-1, m, s-r+(w-km)/z, r, s-r). \tag{3.43}$$

Now, by an application of the Cauchy-Frobenius Theorem we obtain that $\psi_1(n, m, k, w, b) =$

$$\frac{1}{b}\sum_{z|D}\phi(z)\binom{(n-w-z)/z}{(b-z)/z}\sum_{r\geq 0}\binom{\frac{b}{z}}{r}\binom{(k-z)/z}{(b-rz-z)/z} N(m-1, m, s-r+(w-km)/z, r, s-r).$$

□

Let $\mathcal{A}_k^{(m)}(n)$ denote the cyclic analog of $A_k^{(m)}(n)$. Thus, $\mathcal{A}_k^{(m)}(n)$ is the number of equivalence classes under cyclic shift of $n-$dimensional binary vectors containing exactly $k$ isolated $m-$tuples of consecutive ones. Similarly, let $\mathcal{B}_k^{(m)}(n)$ be the cyclic analog of $B_k^{(m)}(n)$. The following are straight forward:

$$\mathcal{A}_k^{(m)}(n) = \sum_{w=0}^{n} \gamma(n, m, k, w) \tag{3.44}$$

$$\mathcal{A}_k^{(m)}(n) = \sum_{b=0}^{\lceil \frac{n}{2} \rceil} \eta(n, m, k, b) \tag{3.45}$$

$$\gamma(n, m, k, w) = \sum_{b=0}^{\lceil \frac{n}{2} \rceil} \psi(n, m, k, w, b) \tag{3.46}$$

$$\eta(n, m, k, b) = \sum_{w=0}^{n} \psi(n, m, k, w, b) \tag{3.47}$$

$$\mathcal{B}_k^{(m)}(n) = \sum_{w=0}^{n} \gamma_1(n, m, k, w) \tag{3.48}$$

$$\mathcal{B}_k^{(m)}(n) = \sum_{b=0}^{\lceil \frac{n}{2} \rceil} \eta_1(n, m, k, b) \tag{3.49}$$

$$\gamma_1(n, m, k, w) = \sum_{b=0}^{\lceil \frac{n}{2} \rceil} \psi_1(n, m, k, w, b) \tag{3.50}$$

and,

$$\eta_1(n, m, k, w) = \sum_{w=0}^{n} \psi_1(n, m, k, w, b). \tag{3.51}$$

Equations (3.44) – (3.51) together with Theorems 3.1 and 3.2 allow us to give a cyclic analog of Theorem 2.3. The resulting explicit formulas for $\gamma(n,m,k,w)$, $\eta(n,m,k,w)$, $\mathcal{A}_k^{(m)}$, $\gamma_1(n,m,k,w)$, $\eta_1(n,m,k,w)$, and $\mathcal{B}_k^{(m)}$ are rather cumbersome and we do not present them here.

# 4 Number of cyclic ordered partitions

It is well known that the formula for the number of integer solutions of equation (1.9) subject to (1.10) has many applications in combinatorial enumeration. These integer solutions are also called *ordered partitions* of $w$ with $b$ parts. Two ordered partitions $(x_1, x_2, \ldots, x_b)$ and $(y_1, y_2, \ldots, y_b)$ are considered to be *cyclically equivalent* if

$$(y_1, y_2, \ldots, y_b) = (x_i, x_{i+1}, \ldots, x_b, x_1, \ldots, x_{i-1})$$

for some $i$. Clearly, this relationship is an equivalence relation according to which all the integer solutions are partitioned into equivalence classes. An equivalence class is called a *cyclic integer solution* of (1.9) subject to (1.10), or a *cyclic ordered partition* of $w$. Let $c(w,b)$ be the number of such cyclic solutions.

Cyclic ordered partitions of $w$ into $b$ parts also have the following combinatorial interpretation. Let there be $w$ points on a circle, and let $b$ bars be placed normally to the circle so that there is at most one bar between any two adjacent points. Two such configurations of points and bars are considered to be equivalent if rotating one produces the other. Then, the number of inequivalent such configurations is clearly also $c(w,b)$. By lemma 3.1 in Section 3 and the Cauchy-Frobenius Theorem we have:

**Theorem 4.1** *If $w$ and $b$ are arbitrary positive integers, and if $D = gcd\,(w, b)$, then $c(w, b)$ is given by:*

$$c(w, b) = \frac{1}{b} \sum_{z|D} \phi(z) \binom{\frac{w}{z} - 1}{\frac{b}{z} - 1}. \tag{4.1}$$

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