5-valent Symmetric Graphs of Order at most 100 *

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Abstract

Let Γ be a finite connected graph and let G be a subgroup of the automorphism group $\mathrm{Aut}\Gamma$ of Γ. Then Γ is said to be G-symmetric, and G is said to be symmetric on Γ, if G is transitive on the set of ordered pairs of adjacent vertices of Γ; Γ is said to be symmetric if $\mathrm{Aut}\Gamma$ is symmetric. It is shown that there are exactly six types of 5-valent G-symmetric graphs of order at most 100 which are not bipartite and on which no subgroup acts regularly. Their orders are 6, 12, 36, 66, 72 and 96.

1 Introduction

Let Γ be a finite connected graph and G be a subgroup of the automorphism group $\mathrm{Aut}\Gamma$ of Γ. Then Γ is said to be G-symmetric, and G is said to be symmetric on Γ, if G is transitive on the set of ordered pairs of adjacent vertices (arc) of Γ; Γ is said to be symmetric if it is $\mathrm{Aut}\Gamma$-symmetric. Note that symmetric graphs (that is those whose automorphism groups act symmetrically) are vertex transitive and hence are regular. The motivation for this paper came from Lorimer [1] about determining all minimal trivalent symmetric graphs of order at most 120. Similar work for 5-valent graphs is more complicated than that for trivalent ones. In the trivalent case the order of a vertex stabilizer has a upper bound that is 48, while in the 5-valent case the order of a vertex stabilizer divides $5 \cdot 3^2 \cdot 2^{17}$ (see [3]). In this paper we give a complete list of 5-valent symmetric graphs which are connected and have order at most 100. In [2] Lorimer gave the following theorem for graphs of prime valency.

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Theorem 1 *(Lorimer) Let Γ be a connected G-symmetric graph of valency p, where p is prime. For each normal subgroup N of one of the following holds:*

(a) Γ is N-symmetric and N is a non-abelian simple group;

(b) N acts regularly on vertices and Γ is a Cayley graph for N;

(c) N has just two orbits on vertices and Γ is bipartite;

(d) $N \cap H = 1$, where H is a vertex stabilizer. N has $r \geq p+1$ orbits on vertices, the natural block graph Γ_N on N-orbits is G/N-symmetric of valency p, and Γ is a topological cover of Γ_N.

In Theorem 1, if G is chosen to be minimal with respect to acting symmetrically, then (d) implies G/N is a non-abelian simple group and from (a) it follows that $G = N$. The purpose of this paper is to investigate cases (a) and (d) in Theorem 1. The results for 5-valent graphs are parallel to Theorem 1 of [1], but some new phenomena appear. In [1] only case (a) happened and no case (d) occurred.

Theorem 2 *Let Γ be a connected 5-valent G-symmetric graph. If Γ is not a bipartite graph and no subgroup of automorphisms acts regularly on $V(\Gamma)$ and if Γ has no more than 100 vertices then Γ is one of the following graphs:*

(a) the complete graph K_6 of order 6 on which $PSL(2,5)$ or $PSL(2,9)$ acts symmetrically;

(b) the icosahedron on which $PSL(2,5)$ acts symmetrically;

(c) a graph of order 96 which is a topological cover of the graph K_6 on which the group $Z_2^4 \cdot A_5$ acts symmetrically and the automorphism group of the block graph K_6 is $PSL(2,5)$;

(d) the graph $L_2(9)^5_{72}$ of order 72 on which $PSL(2,9)$ acts symmetrically;

(e) the graph $L_2(9)^5_{36}$ of order 36 on which $PSL(2,9)$ acts symmetrically;

(f) a graph of order 96 which is a topological cover of graph the K_6 on which the group $Z_2^4 \cdot A_6$ acts symmetrically and the automorphism group of the block graph K_6 is $PSL(2,9)$;

(g) the graph $L_2(11)^5_{66}$ of order 66 on which $PSL(2,11)$ acts symmetrically.

In section 2, we quote some lemmas which will be used later. In section 3, Theorem 2 is proved. For all the group-theoretic concepts not defined here we refer the reader to [6, 7].

2 Preliminary Lemmas

As a generalization of Cayley digraphs, Sabidussi [10] gave another construction of vertex-transitive digraphs using groups; it is known as a Sabidussi coset graph.

Definition 2.1 *Let G be a finite group and H be a subgroup of G. Let D be a union of several double cosets of the form HgH, not containing the subgroup H. We define the Sabidussi coset digraph $\Gamma = \mathrm{Sab}(G,H,D)$ of G with respect to H and D by*

$$\begin{aligned} V(\Gamma) &= \{gH \mid g \in G\}, \\ E(\Gamma) &= \{(gH, gdH) \mid g \in G, d \in D\}. \end{aligned}$$

Note that we do not consider multigraphs, so if $gdH = gd_1H$, the edges (gH, gdH) and (gH, gd_1H) are viewed as equal.

The following obvious facts are basic for the Sabidussi coset graph.

Lemma 2.2 *Let $\Gamma = \mathrm{Sab}(G, H, D)$ be the Sabidussi coset digraph of G with respect to H and D. Then*

(1) Γ is a well-defined digraph with in-degree and out-degree $|D : H|$.

(2) $\mathrm{Aut}\Gamma$ contains G by left multiplication, so Γ is vertex-transitive. For a vertex gH, the stabilizer in G is gHg^{-1}.

(3) Γ is connected if and only if $G = \langle D \rangle$.

(4) Γ is undirected if and only if $D^{-1} = D$.

(5) Γ is G-symmetric if and only if $D = Hg_iH$ is a single double coset.

Note that Cayley graphs are the special case of Sabidussi coset graphs with $H = 1$.

Any vertex-transitive graph (digraph) is a Sabidussi coset digraph. In fact, given a vertex-transitive graph (digraph) Γ and a vertex $v \in V(\Gamma)$, take $G = \mathrm{Aut}\Gamma$, $H = G_v$, and $D = \{g \in G \mid v_g \in \Gamma_1(v)\}$, then D is a union of several double cosets of the form HgH with $D \cap H = \emptyset$ and $\Gamma \cong \mathrm{Sab}(G, H, D)$.

So, in theory, if we knew all groups and their subgroup structure, then we would know all vertex-transitive graphs (digraphs) and symmetric graphs.

Using Lemma 2.2, we can prove following lemma of [2] for our graphs.

Lemma 2.3 *The group G acts symmetrically on a 5-valent connected graph Γ if and only if it has a subgroup H and member a such that*

(a) $a^2 \in H$,

(b) $H \cap aHa^{-1}$ has index 5 in H,

(c) G is generated by HaH.

Lemma 2.4 *In Lemma 2.3, a must be an element of G of even order.*

Proof If the order of a is an odd number, say k, then k is relatively prime to 2. Thus there exists integers m and n such that $mk + 2n = 1$. It follows that

$$a^{mk}a^{2n} = a^{mk+2n} = a \in H$$

which contradicts the assumption $a \notin H$. □

For convenience we state some well known results which will be used later

Lemma 2.5 *(Weiss [3]) The order of a vertex stabilizer divides $5 \cdot 3^2 \cdot 2^{17}$.*

Lemma 2.6 *(Gaschütz) Let N be an abelian normal subgroup of G, suppose $N \leq B \leq G$ and that the order of N and the index of B in G are relatively prime. If N has a complement in B then it also has a complement in G.*

Lemma 2.7 *(see [5])*

For every $n \geq 5$ the alternating group A_n can be generated by an involution a and another suitable element b:

(1) $a = (1, 2)(n-1, n), b = (1, 2, \cdots, n-1)$ if n is even;

(2) $a = (1, n)(2, n-1), b = (1, 2, \cdots, n-2)$ if n is odd.

3 The Proof of Theorem 2

Proof of Theorem 2: Let Γ be a graph which satisfies the hypotheses of Theorem 2: thus Γ be a 5-valent symmetric graph of order at most 100, which is not a bipartite graph. Let G be a group which acts symmetrically on Γ and suppose that G has no proper subgroup with this property and no subgroup acts on Γ regularly.

Let α be a fixed vertex of Γ and let H be its stabilizer in G. Let $\beta_i, i = 1,2,3,4,5$ be the vertices of Γ adjacent to α and let $a_i \in G, i = 1,2,3,4,5$ have the properties $a_i(\alpha) = \beta_i$ and $a_i^2 \in H, i = 1,2,3,4,5$. Let N be a maximal normal subgroup of G. Hence N acts semi-regularly on Γ (i.e.$N \cap H = 1$) and G/N is a simple group.

The notation established in last two paragraphs will be maintained throughout this section.

The proof of Theorem 2 is organized into foreteen Lemmas. First since Γ is not a bipartite graph, it follows that H and N are subject to the conditions in the following lemma.

Lemma 3.1 *(a) HN has even index in G;*
(b) G has no subgroup of index 2 which contains H.

Proof See [1]. □

In order to give a completed list of 5-valent graphs of order at most 100, we search for simple groups G/N satisfying the following hypotheses.

Hypotheses 3.2 *Let G/N be a simple group of order at most 589,824,000 such that there exists a subgroup H satisfying the following conditions:*
(1) 5 is the exact power of 5 which divides $|H|$;
(2) H has even index at most 100 in G;
(3) H satisfies Lemma 2.5;
(4) H satisfies Lemma 3.1.

Lemma 3.3 *If Γ satisfies the conditions of Theorem 2 then G must satisfy Hypotheses 3.2.*

Proof By Lemma 2.5, the order of H is at most $5 \cdot 3^2 \cdot 2^{17} = 5,898,240$. As Γ has at most 100 vertices and it is defined by left cosets of H, G has order at most 589,824,000 and so does G/N. So we have all possible 5-valent graphs which come from left coset graphs $\Gamma = \mathrm{Sab}(G, H, D)$. However these simple groups must be subject to the relations of Lemma 2.3, since Γ is a symmetric graph. As Γ is G-symmetric, H acts transitively on the set $\Gamma_1(\alpha)$ of neighbours of vertex α, and hence the order of H is divisible by 5. (2) and (4) hold obviously. □

Lemma 3.4 *The possibilities for G/N, H and $|N|$ are as in Table 1.*

Table 1 The possibilities for G/N, H, $|N|$

No	G/N	H	$\|N\|$	*No*	G/N	H	$\|N\|$
(1)	A_7	A_5	≤ 2	*(9)*	$PSL(2,9)$	Z_5	1
(2)	A_8	$A_5 \times 3:2$	1	*(10)*	$PSL(2,5)$	Z_5	≤ 8
(3)	A_8	S_6	≤ 3	*(11)*	$PSL(2,5)$	D_{10}	≤ 16
(4)	$PSL(2,16)$	A_5	1	*(12)*	M_{11}	S_5	1
(5)	$PSL(3,4)$	A_6	1	*(13)*	M_{11}	A_6	≤ 4
(6)	$PSL(2,11)$	D_{10}	1	*(14)*	M_{12}	$M_{10}:2$	1
(7)	$PSL(2,9)$	A_5	≤ 16	*(15)*	$U_4(2)$	S_6	≤ 2
(8)	$PSL(2,9)$	D_{10}	≤ 2	*(16)*	$U_4(2)$	A_6	1

Proof According to the Atlas [6, p240], the number of simple group of order at most 589,824,000 is 86. If we arrange them according to their order , the last one is $PSL(3,13)$. First we exclude 27 simple groups of order at most 589,824,000 which have no divisor of 5 by Hypotheses 3.2 (1). The second 18 simple groups which are excluded are those whose order has 5 as a divisor but the smallest index of a proper subgroup is at least 101, including 4 members of the family of $PSL(2,q)$. The remainer we list in table 2 except the family of $PSL(2,q)$. In table 2, M means the maximal subgroup whose index is at most 100, and we exclude directly those not satisfying the Hypotheses 3.2.

So the simple groups of order at most 589,824,000 satisying Hypotheses 3.2, are $PSL(2,5)$, $PSL(2,9)$, $PSL(2,11)$, A_7, $PSL(2,16)$, $PSL(3,4)$, M_{11}, M_{12}, $U_4(2)$, and A_8. Applying the following inequality

$$|N| \cdot |G/N| \leq 100 \cdot |H|, \tag{1}$$

elementary calculations lead to table 1. □

Table 2 Excluding groups which do not satisfy Hypotheses 3.2

No	G	order	M	index	exclude	not hold
1	M_{24}	$2^{10}.3^3.5.7.11.23$	M_{23}	24	yes	$H3.2$ (3)
2(a)	A_{12}	$2^9.3^5.5.7.11$	A_{11}	12	yes	$H3.2$ (3)
2(b)	A_{12}	$2^9.3^5.5.7.11$	S_{10}	66	yes	$H3.2$ (3)
3	A_{12}	$2^9.3^2.5^3.7.11$	M_{22}	100	yes	$H3.2$ (3)
4	$PSL(3,9)$	$2^7.3^6.5.7.13$	$GL(2,9)$	91	yes	$H3.2$ (2)
5(a)	A_{11}	$2^7.3^4.5^2.7.11$	A_{10}	11	yes	$H3.2$ (3)
5(b)	A_{11}	$2^7.3^4.5^2.7.11$	S_9	55	yes	$H3.2$ (3)
6	M_{23}	$2^7.3^2.5.7.11.23$	M_{22}	23	yes	$H3.2$ (3)
7	$PSL(5,2)$	$2^10.3^2.5.7.31$	$2^4:PSL(4,2)$	31	yes	$H3.2$ (3)
8	$PSL(4,3)$	$2^7.3^6.5.13$	$3^3:PSL(3,3)$	40	yes	$H3.2$ (3)
9(a)	A_{10}	$2^7.3^4.5^2.7$	A_9	10	yes	$H3.2$ (3)
9(b)	A_{10}	$2^7.3^4.5^2.7$	S_8	45	yes	$H3.2$ (3)
10(a)	$S_6(2)$	$2^9.3^4.5.7$	$U(2):2$	28	yes	$H3.2$ (3)

Table 2 Excluding groups which do not satisfy Hypotheses 3.2 (continuation)

10(b)	$S_6(2)$	$2^9.3^4.5.7$	S_8	36	yes	$H3.2$ (3)
10(c)	$S_6(2)$	$2^9.3^4.5.7$	$2^5 : S_6$	63	yes	$H3.2$ (2)
11	$S_4(4)$	$2^8.3^2.5^2.17$	$2^6 : (3 \times A_5)$	85	yes	$H3.2$ (2)
12	J_2	$2^7.3^3.5^2.7$	$U_3(3)$	100	yes	$H3.2$ (3)
13(a)	M_{22}	$2^7.3^2.5.7.11$	$PSL(3,4)$	22	yes	$H3.2$ (3)
13(b)	M_{22}	$2^7.3^2.5.7.11$	$2^4. : A_6$	77	yes	$H3.2$ (3)
14	$PSL(3,5)$	$2^5.3.3^5.31$	$5^2 : GL_2(5)$	31	yes	$H3.2$ (3)
15(a)	A_9	$2^6.3^4.5.7$	A_8	9	yes	$H3.2$ (2)
15(b)	A_9	$2^6.3^4.5.7$	S_7	36	yes	$H3.2$ (3)
15(c)	A_9	$2^6.3^4.5.7$	$(A_6 \times 3) : 2$	84	yes	$H3.2$ (3)
16	$U_3(5)$	$2^4.3^2.5^3.7$	A_7	50	yes	$H3.2$ (3)
17(a)	M_{12}	$2^6.3^3.5.11$	M_{11}	12	yes	$H3.2$ (3)
17(b)	M_{12}	$2^6.3^3.5.11$	$M_{10} : 2$	66	no	
18	$U_3(4)$	$2^6.3.5^2.13$	$2^{2+4} : 15$	65	yes	$H3.2$ (3)
19	$Sz(8)$	$2^6.5.7.13$	$2^{3+3} : 7$	65	yes	$H3.2$ (3)
20(a)	$U_4(2)$	$2^6.3^4.5$	$2^4 : A_5$	27	yes	$H3.2$ (2)
20(b)	$U_4(2)$	$2^6.3^4.5$	S_6	36	no	
20(c)	$U_4(2)$	$2^6.3^4.5$	$3^{1+2}_+ : 2A_4$	40	yes	$H3.2$ (1)
20(d)	$U_4(2)$	$2^6.3^4.5$	$3^3 : S_4$	40	yes	$H3.2$ (1)
20(e)	$U_4(2)$	$2^6.3^4.5$	$2.(A_4 \times A_4).2$	45	yes	$H3.2$ (1)
21(a)	$PSL(3,4)$	$2^6.3^2.5.7$	$2^4 : A_5$	21	yes	$H3.2$ (3)
21(b)	$PSL(3,4)$	$2^6.3^2.5.7$	A_6	56	no	
22(a)	A_8	$2^6.3^2.5.7$	A_7	8	no	
22(b)	A_8	$2^6.3^2.5.7$	$2^3 : PSL(3,2)$	15	yes	$H3.2$ (1)
22(c)	A_8	$2^6.3^2.5.7$	S_6	28	no	
22(d)	A_8	$2^6.3^2.5.7$	$2^4 : (S_3 \times S_3)$	35	yes	$H3.2$ (1)
22(e)	A_8	$2^6.3^2.5.7$	$(A_5 \times 3) : 2$	56	no	
23(a)	M_{11}	$2^4.3^2.5.11$	$A_6.2$	11	no	
23(b)	M_{11}	$2^4.3^2.5.11$	$PSL(2,11)$	12	yes	$H3.2$ (3)
23(c)	M_{11}	$2^4.3^2.5.11$	S_5	11	no	
23(d)	M_{11}	$2^4.3^2.5.11$	$M_8 : S_3$	11	yes	$H3.2$ (3)
24(a)	A_7	$2^3.3^2.5.7$	A_6	5	no	
24(b)	A_7	$2^3.3^2.5.7$	$PSL(2,7)$	15	yes	$H3.2$ (1)
24(c)	A_7	$2^3.3^2.5.7$	$(A_4 \times 3) : 2$	35	yes	$H3.2$ (1)
24(d)	A_8	$2^6.3^2.5.7$	S_5	21	no	

Lemma 3.5 *Assume $G/N \cong PSL(2,5)$.*

(a) If $H \cong Z_5$ and $|N| \leq 8$, then $N = 1$ and Γ is the graph of the vertices and edges of the icosahedron, which has order 12:

(b) if $H \cong D_{10}$ and $|N| \leq 15$, then $N = 1$ and Γ is the complete graph K_6 of order 6.

(c) if $H \cong D_{10}$ and $|N| = 16$, then $N = Z_2^4$, the block graph $\Gamma_N = K_6$, Γ is a topological cover of Γ_N and the order of Γ is 96; G is an extension of Z_2^4 by A_5.

Proof First we give preliminary facts about A_5 which will be used later. We know A_5 can be generated by an element of order 5 with an involution. In fact, without loss of generality, take $h = (12345)$ and take any involution a from (15)(23), (12)(34), (23)(45), (12)(45) and (34)(15). Then it is easy to check that a and h generate A_5 according to the relations

$$a^2 = h^5 = (ah)^3 = 1.$$

On other hand, the element h of order 5 is contained in just one subgroup isomorphic to D_{10}: if h=(12345) this group is $\{1, (12345), (13524), (14253), (15432), (15)(24), (23)(14), (45)(13), (12)(35), (34)(25)\}$.

If $G = A_5$ and $H = \langle h\rangle \cong Z_5$, then $G = \langle a, H\rangle$ is discussed as above. By Lemma 2.3, Γ is the icosahedron, which is defined on $\{gH|g \in G\}$. If $G = A_5$ and $H \cong D_{10}$ as above, then $G = \langle a, H\rangle$ and Γ is the complete graph K_6, which is defined on $\{gH|g \in G\}$ as in Definition 2.1.

Suppose, now, that G is as in Lemma 3.4 (10) or (11), that is $G/N \cong A_5$, $H \cong Z_5$, $|N| \leq 8$ or $H \cong D_{10}$, $|N| \leq 16$. Let C be the centralizer of N in G. Since N is a normal subgroup of G, so is C. As G/N is simple, either $C \leq N$ or $CN = G$. By Lemma 3.4, $|N| \leq 16$ if $H \cong D_{10}$, so we shall treat two subcases $|N| \leq 15$ and $|N| \leq 16$ separately.

Subcase 1. $|N| \leq 15$.

In this case we shall prove $N = 1$. We prove it in six steps.

(i) First we prove $CN = G$. If not, then $C \leq N$, and G/C is an automorphism group of N. It is impossible for G/C to have A_5 as a factor group since $|N| \leq 15$. Thus $CN = G$.

(ii) We claim that $C = G$, and hence N is the center of G. If not, $C \neq G$. By Theorem 1 and the assumption, no normal sugroup acts regularly on Γ, $H \cap C = 1$ and $HC \neq G$. Since G is generated by HaH, $a \notin HC$. As $G/C = NC/C \cong N/N \cap C$, G/C has order at most 15.

Suppose that G is as in Lemma 3.4 (10). Since $H \cong Z_5$ and $|N| \leq 8$, $|G/C| \leq 8$. As $H \cap C = 1$, HC/C is a proper subgroup of G/C of order 5, and this contradicts $|G/C| \leq 8$.

Suppose that G is as in Lemma 3.4 (11). Since $H \cong D_{10}$ and $|N| \leq 15$, so $|G/C| \leq 15$. As $H \cap C = 1$, the proper subgroup HC/C of G/C has order 10, and this contradicts $|G/C| \leq 15$. These considerations were based on the assumption $C \neq G$. Therefore, $G = C$ and N is the center of G, establishing the claim. In particular, N is abelian.

(iii) Now we prove $G = \langle a, h\rangle$, where h is an element of of order 5 of H. Let $M = \langle a, h\rangle$. Since $G = \langle a, H\rangle$, if $H \cong Z_5$ then $M = G$. Suppose that $H \cong D_{10}$. As $G/N \cong A_5$ and $G = \langle a, H\rangle$, $G/N = \langle aN, HN\rangle$ according to the relations

$$(aN)^2 = (hN)^5 = (ahN)^3 = N.$$

G/N is not generated by H. Thus $a \notin N$. As $a^2 \in H$ and $H \cap aHa^{-1}$ has index 5 in H, a is either an involution or it has order 4. As A_5 has no element of order 4 it must be that $a^2 = 1$. Let h be an element of order 5 in H. Thus by $H \cap N = 1$, we get $a^2 = h^5 = 1, (ah)^3 = z \in N$.

As $a^2 = 1$, a normalizes $H \cap aHa^{-1} = \{1, b\}$ and hence $ab = ba$. If $H \cong D_{10}$, $hb = bh^{-1}$. Thus b normalizes M. As G is generated by H and a, $G = M \cup bM$. Since $|M \cap H| = 5$ and $|H| = 10$, then $|M : M \cap H| = |G : H|$. Since $M = \langle a, h \rangle = \langle a, M \cap H \rangle$, by Lemma 2.3 we conclude that Γ is M-symmetric which contradicts the minimal property of G. Hence, G is generated a by and h which satisfy the relations $a^2 = h^5 = 1, (ah)^3 = z \in N$, establishing the claim.

(*iv*) Let P be a Sylow p-subgroup of N for some $p \neq 2, 3$. We claim that $P = 1$.

As N is the center of G, P is a normal subgroup of G and $Q = P \times \langle h \rangle$ is a subgroup of G. Since $(|G : Q|, p) = 1$ and P has a complement in Q, it follows from Lemma 2.5 that P has a complement P_1 in G. As P lies in the center of G, $G = P \times P_1$. If $p \neq 5$, P_1 contains Sylow 2-subgroups and Sylow 5-subgroups of G and hence contains a and h which generate G. Thus $G = P_1$ and $P = 1$.

Suppose $p = 5$ and $P \neq 1$. Then P_1 contains all the Sylow 2-subgroup of G and hence contains a. Since $G = \langle a, h \rangle = \langle P_1, h \rangle$, so

$$G = P_1 \cup hP_1 \cup h^2 P_1 \cup h^3 P_1 \cup h^4 P_1.$$

If $H = \langle h \rangle \cong Z_5$ then P_1 is a normal complement of H in G and so acts regularly on Γ, contrary to the hypothesis. Hence $H \cong D_{10}$ and b lies in P_1. Since P_1 is a normal subgroup of G, $hP_1 = P_1 h$. It follows that $bhb^{-1}P_1 = bhP_1 = bP_1 h = P_1 h = hP_1$. That is $bhb^{-1} \in hP_1$. However, $bhb^{-1} = h^{-1} \in h^4 P_1$, a contradiction. Thus the Sylow 5-subgroup of N is trivial as are the Sylow p-subgroups for $p \neq 2, 3$.

(*v*) Next, let P be a Sylow 3-subgroup of N. We shall prove $P = 1$.

Let Q be a Sylow 3-subgroup of G which contains P. Then QN/N is a Sylow 3-subgroup of G/N. Hence QN/N is cyclic of order 3 and there is an involution of G/N which maps each element of QN/N onto its inverse. Let x be a member of $Q - P$. Then G contains a member y such that $yxy^{-1} = x^{-1}n$ for some $n \in N$. As $x^3 \in N$, so $x^3 = yx^3y^{-1}$. Thus $x^3 = yx^3y^{-1} = (yxy^{-1})^3 = x^{-3}n^3$ and $1 = x^{-6}n^3$. Let $u = x^{-1}$. We have $(u^2 n)^3 = 1$. The subgroup generated by $u^2 n$ is a complement of P in Q. Thus P has a complement P_1 in G by Lemma 2.5. As P lies in the center of G, G is the direct product of P and P_1. As the order of P is a power of 3, every element of order 2 or 5 lies in P_1. In particular, $a, h \in P_1$ and, as G is generated by a and h, $G = P_1$ and $P = 1$.

(*vi*) Since no other prime is a divisor of the order of N, N must be a 2-group. Now we prove $N = 1$.

Let Q be a subgroup of N which has index 2 in N. Then the members aQ and hQ of G/Q satisfy the relations $(aQ)^2 = (hQ)^5 = (ahQ)^6 = Q$. Put $kQ = (ahQ)^3$. Thus $(kQ)^2 = Q$ and $(akQ)^2 = (hQ)^5 = (akhQ)^3 = Q$. That is, $\langle akQ, hQ \rangle = P_1/Q \cong A_5$. As N/Q is the centre of G/Q, $G/Q = P_1/Q \times N/Q$. Since N/Q has order 2, P_1/Q is a subgroup of index 2 and P_1 is a subgroup of G of index 2. Hence, by lemma 3.1(b), H is not a subgroup P_1. This forces $H \cong D_{10}$ and $G = P_1 \cup bP_1$. In these circumstances $P_1 \cap H$ has the same number of cosets in P_1 as H has in G, i.e. P_1 acts transitively on the vertices of Γ. As $h \in P_1$ it also acts symmetrically which is not possible because of the minimal property of G. Hence N has no subgroup Q of index 2 and as N is 2-group it follows that $N = 1$, $G \cong A_5$, and Γ is one of the two 5-valent graphs as conclusions (a) and (b) of this Lemma on which A_5 acts symmetrically.

Subcase 2. H isomorphic to D_{10} and $|N| = 16$.

In this case, if $K = C_G(N) = G$, we have the conclusion, as discussed above. So we assume that $K \leq N$. Since N is a subgroup of order 2^4, by [5,Th 5.3] the order of Aut(N) divides

$$p^{d(n-d)}(p^d - 1)(p^d - p) \cdots (p^d - p^{d-1}), \tag{2}$$

where d is the rank of the p-group and $p = 2$. As the order of G/N divides that of G/K and $G/K \leq \mathrm{Aut}(N)$, the order of A_5 divides (2). It follows from $p = 2$ that $d = 4$. Thus $\Phi(N) = 1$. Hence $N \cong Z_2^4$ and $N = K$.

As $G/N \cong A_5$, G/N is generated by aN and hN, subject to the relations $(aN)^2 = (hN)^5 = (ahN)^3 = N$. As $H \cap N = 1$ and $a^2, h \in H$, thus $a^2 = h^5 = 1$, $(ah)^3 = n \in N$. Because N is an elementary abelian 2-group, we get $a^2 = h^5 = (ah)^6 = n^2 = 1$. As the order of N is 16, the length of the orbits of N on Γ is 16. By the assumption that the order Γ is at most 100, it follows from Theorem 1.(d) that the number of orbits of N is 6. So the block graph $\Gamma_N \cong K_6$. Since $H \cap N = 1$, Γ is a topological cover of Γ_N and the order of Γ is 96. (We recall that Γ is said to be a topological cover of its block graph Γ_N if, whenever two vertex xHN and yHN are adjacent in Γ_N, each vertex in xHN is adjacent in Γ to exactly one vertex in yHN). This completes the proof. □

Lemma 3.6 *G/N is not isomorphic to A_7, i.e. case (1) of Lemma 3.4 does not occur.*

Proof By Lemma 3.4 (1), $H \cong A_5$ and $|N| \leq 2$. To prove this lemma, it suffices to prove that there is no $a \in G$ such that quotient group $G/N = \langle aN, HN \rangle \cong A_7$ subject to the relations of Lemma 2.3. For convenience we use $A_7, \overline{a}, \overline{H}$ in instead of $G/N, aN, HN$ respectively in the rest of our discussion, and it may be supposed, without loss of generality, that we take the members of $H \cong A_5$ as the even permutations on the set $\{1,2,3,4,5\}$. Similarly we take the members of A_7 and A_4 as the even permutations on the sets $\{1,2,3,4,5,6,7\}$ and $\{1,2,3,4\}$ respectively.

If not, choose $\overline{a} \in A_7$ such that $\overline{H} \cap \overline{H}^{\overline{a}}$ has index 5 in $\overline{H}$ and $\langle \overline{a}, \overline{H} \rangle = A_7$. Hence $\overline{H} \cap \overline{H}^{\overline{a}} \cong A_4$. If $\overline{a}$ is involution, then $\overline{a}$ fixes $\overline{H} \cap \overline{H}^{\overline{a}}$ and thus fixes the unique Sylow 2-subgroup Q of $\overline{H} \cap \overline{H}^{\overline{a}}$. As $\overline{H} \cap \overline{H}^{\overline{a}}$ has no Sylow 5-subgroup, $\overline{a}$ must interchanges 5 and 6 or 5 and 7. Thus $\overline{a} = (i,j)(5,6)$ or $(i,j)(5,7)$, where $i, j \in \{1,2,3,4\}$. However $A_7 \neq \langle \overline{a}, \overline{H} \rangle$, since the group $\langle \overline{a}, \overline{H} \rangle$ is a permutation group of a six letter set. So $\overline{a}$ is an element of order 4. As $\overline{a}^2 \in \overline{H}$, $\overline{a}$ fixes $\overline{H} \cap \overline{a}\overline{H}\overline{a}^{-1}$. So $\overline{a}$ induces an automorphism on $\overline{H} \cap \overline{H}^{\overline{a}}$ and thus fixes the Sylow 2-subgroup Q of $\overline{H} \cap \overline{H}^{\overline{a}}$. Thus $\overline{a} = (1,2,3,4)(5,6)$ or $a = (1,2,3,4)(5,7)$. Therefore $\langle a, H \rangle$ is a permutation group of a six letter set, contrary to our assumption $A_7 = \langle \overline{a}, \overline{H} \rangle$. This proves the lemma. □

Lemma 3.7 *G is not isomorphic to A_8 i.e. case (2) of Lemma 3.4 does not occur.*

Proof By Lemma 3.4 (2), $H \cong A_5 \times 3 : 2$. To prove this lemma it suffices to prove that there is no element a such that $G = \langle a, H \rangle$ subject to the relations of Lemma

2.3. If not, choose $a \in G$ such that $H \cap H^a$ has index 5 in H and $G = \langle a, H \rangle$. Hence $H \cap H^a \cong A_4 \times 3 : 2$. As $a^2 \in H$, it follows that a fixes $H \cap H^a$. Thus $\langle a, H \cap H^a \rangle \cong A_4 \times 3 : 4$. However A_8 has no subgroup isomorphic to $A_4 \times 3 : 4$, this is a contradiction. This proves the lemma. □

Lemma 3.8 *Assume $G/N \cong PSL(2,9)$. Then*

(a) if $H \cong Z_5$ then $\Gamma = L_2(9)^5_{72}$ is the graph of order 72;

(b) if $H \cong D_{10}$ and $|N| \leq 2$, then $N = 1$ and $\Gamma = L_2(9)^5_{36}$ is the graph of order 36;

(c) if $H \cong A_5$ and $|N| \leq 15$, then $N = 1$ and Γ is the complete graph K_6 of order 6;

(d) if $H \cong A_5$ and $|N| = 16$, then $N \cong Z_2^4$, the block graph $\Gamma_N = K_6$, Γ is a topological cover of Γ_N and the order of Γ is 96; G is the extension of Z_2^4 by A_6.

Proof It is convenient to use the isomorphism $PSL(2,9) \cong A_6$ and take its members as the even permutations on the set $\{1,2,3,4,5,6,\}$. A_6 contains two conjugacy classes of subgroups $H \cong Z_5$, one generated by (12345), and the other by (12346). As these subgroups are conjugates within the automorphism group of A_6 it may be supposed, without loss of generality, that H is the first. If $h = (12345)$, it follows from Lemma 2.4 that there is an element $a = (12)(56)$ with h generating A_6.

(a). Now consider the possibility described in case (9) of Lemma 3.4, that is, $G \cong A_6$ and $H \cong Z_5$. Choose a and h as above and set $H = \langle h \rangle (\cong Z_5)$. Then $H \cap H^a = 1$ and $G = \langle a, h \rangle = \langle a, H \rangle$. As A_6 has order 360, the subgroup H with the relevant element a defines a graph of order 72 on which A_6 acts symmetrically by Lemma 2.3, which we denote by $L_2(9)^5_{72}$.

(b). Now we prove case (b), that is, $G/N \cong A_6$, $N \leq 2$ and $H \cong D_{10}$.

First consider the case $N = 1$, that is, $G \cong A_6$. Choose $a = (1243)(56)$, $h = (12345)$, then $\langle a, h \rangle = A_6$ according to the relations

$$a^4 = h^5 = (ah)^5 = (a^2h)^2 = 1.$$

If $\langle a, h \rangle \neq A_6$, then $\langle a, h \rangle \leq M$ a maximal subgroup of A_6. However A_6 has no maximal subgroup M which contains both elements of order 5 and elements of order 4. This contradiction shows $G = \langle a, h \rangle$ as claimed. As A_6 has order 360, the subgroup $H \cong D_{10}$ with the relevant element a define graphs of order 36, which we denote by $L_2(9)^5_{36}$, on which A_6 acts symmetrically.

Now consider the possibility in Lemma 3.4 (8). In this case $G/N \cong A_6$, $|N| \leq 2$ and $H \cong D_{10}$. We claim $N = 1$. If not, $|N| = 2, G = \langle a, H \rangle$ subject to the relations of Lemma 2.3. As $H \cap aHa^{-1}$ has index 5 in H, it is easy to prove a is not an involution. Thus a^2 must be an involution of H. Hence there exists $h \in H$ such that $\langle aN, hN \rangle = G/N$ according to the relations

$$(aN)^4 = (hN)^5 = (ahN)^5 = (a^2hN)^2 = N.$$

Since $a^2h \in H$, so $a^4 = h^5 = (a^2h)^2 = 1$. Suppose that $(ah)^5 = z \in N, z^2 = 1$. Since $N = 2$ and G/N is a simple group of order 360, N is the center of G. Then

$$(az)^4 = h^5 = (azh)^5 = ((az)^2h)^2 = 1.$$

So $\langle az, h\rangle \cong A_6$. As N is the center of G it must be that $G = M \times N$, a direct product. As $|N| = 2$, then $|G : M| = 2$. As h and a^2 both lie in M, it shows $H \leq M$ and contradicts Lemma 3.1 (b). It follows that $N = 1$ as claimed and hence case (b) is proved.

In case(c), that is $G/N \cong A_6$, $H \cong A_5$ and $|N| \leq 15$. By the same method as Lemma 3.5 (i), (ii) we can prove that N is the center of G. Now we prove $N = 1$ by following four steps.

(i). Let P be a Sylow p-subgroup of N for some $p \neq 2, 3, 5$. As N is the center of G, P is a normal subgroup of G. By the Schur-Zassenhaus theorem, P has a complement Q in G. Since Q contains Sylow 2-subgroups and Sylow 5-subgroups of G and hence contains a and A_5 which generate G, thus $G = Q$ and $P = 1$.

(ii). Let P be a Sylow 5-subgroup of N. As N is the center of G, P is a normal subgroup of G and $P_1 = P \times \langle h\rangle$ is a Sylow 5-subgroup of G, where $h \in H$ is as above. Since $(|G : P_1|, 5) = 1$ and P has a complement in P_1, it follows from Lemma 2.5 that P has a complement Q in G. As P lies in the center of G, $G = P \times Q$. Since Q contains Sylow 2-subgroups and Sylow 3-subgroups of G, and since $H \cong A_5$ can also be generated by an element of order 2 with an element of order 3, it follows that $Q \geq \langle a, H\rangle = G$. Thus $G = Q$ and $P = 1$.

(iii). Let P be a Sylow 3-subgroug of N. Choose $t \in H$ and $s \in G$ such that $\langle tN, sN\rangle$ is a Sylow 3-subgroup of $G/N \cong A_6$. Thus we have $(sN)^3 = (tN)^3 = N$ and hence $s^3, t^3 \in N$. Since $H \cap N = 1$, we deduce $t^3 = 1$. On the other hand, G has an element y of even order such that $ysy^{-1} = t^{-1}n$ for some $n \in N$. Since $s^3 \in N$, $s^3 = ys^3y^{-1} = (ysy^{-1})^3 = (t^{-1})^3n^3 = n^3$. So $(sn^{-1})^3 = 1$ and $\langle sn^{-1}, t\rangle \times P$ is a Sylow 3-subgroup of G. Using Lemma 2.5, P has a normal complement Q. Thus every element of order 2 or 5 lies in Q and hence $Q \geq \langle a, H\rangle = G$. It follows that $P = 1$.

(v). As no other prime is a divisor of the order of N, N must be a 2-group. Let Q be a subgroup of N which has index 2 in N. Since $G/N \cong A_6$, there exist $a, h \in G$ such that $(aN)^4 = (hN)^5 = (ahN)^5 = (a^2hN)^2 = N$ as in (b). So $a^4, h^5, (a^2h)^2, (ah)^5 \in N$. Since $a^2, h, a^h \in H$ and $H \cap N = 1$, it follows that $a^4 = h^5 = (a^2h)^2 = 1$ and $(ah)^{10} \in Q$. Therefore

$$(aQ)^4 = (hQ)^5 = (ahQ)^{10} = (a^2hQ)^2 = Q.$$

Set $zQ = (ahQ)^5$. Then $(zQ)^2 = Q$ and

$$(azQ)^4 = (hQ)^5 = (azhQ)^5 = ((az)^2hQ)^2 = Q.$$

Thus $M/Q = \langle azQ, hQ\rangle \cong A_6$. Hence $G/Q = M/Q \times N/Q$. Since $|N/Q| = 2$, $|G : M| = 2$. Hence, by Lemma 3.1 (b) H is not a subgroup of M. It shows that H contains an involution b such that $G = M \cup bM$ and $|M : M \cap H| = |G : H|$. That is, M acts transitively on the vertices of Γ. As $h \in M$, Γ is M-symmetric. This contradicts the minimal property of G. Hence N has no subgroup Q of index 2 and as N is a 2-group it follows that $N = 1$.

Now choose $a = (12)(56)$ and $H \cong A_5$. As in Lemma 3.6, we have $G = \langle a, H\rangle$ and $H \cap H^a$ has index 5 in H. It follows that Γ defined by $\{gH | g \in G\}$ is the complete graph K_6 of order 6 on which A_6 acts symmetrically.

(d) if $H \cong A_5$ and $|N| = 16$, then $N = Z_2^4$ by the method of Subcase 2 of Lemma 3.5 and it follows that Γ is a topological cover of $\Gamma_N \cong K_6$ and the order of Γ is 96 and G is the extension of Z_2^4 by A_6. □

Lemma 3.9 *If G is isomorphic to $PSL(2,11)$ then $\Gamma \cong L_2(11)^5_{66}$ which is defined in [9].*

Proof By Lemma 3.4 (6), $G \cong PSL(2,11)$, $H \cong D_{10}$. In this case Γ is the unique vertex-primitive graph of order 66 which is determined in Lemma 4.4 of [9] and so we omit the direct check. We denote it by $L_2(11)^5_{66}$. □

Lemma 3.10 *G is not isomorphic to $PSL(2,16)$ i.e. case (4) of Lemma 3.4 does not occur.*

Proof By Lemma 3.4 (4), $H \cong A_5$. To prove this lemma it suffices to prove that there is no element a such that $G = \langle a, H\rangle$ subject to the relations of Lemma 2.3. If not, choose $a \in G$ such that $H \cap H^a$ has index 5 in H and hence $H \cap H^a \cong A_4$. As $PSL(2,16)$ has no element of order 4, a must be an involution. As a fixes $H \cap H^a$, $\langle a, H \cap H^a\rangle \cong S_4$. This contradicts the fact $PSL(2,16)$ has no subgroup isomorphic to S_4. The proof is complete. □

Lemma 3.11 *If $H \cong A_6$ or $H \cong S_6$, then Γ is not a 5-valent graph.*

Proof If Γ is a 5-valent graph then $|H : H \cap H^a| = 5$ where a with H generates G. Thus if $H \cong A_6$, then $|H \cap H^a| = 72$. However A_6 has no subgroup of order 72, so $H \cong A_6$ is impossible. Similarly S_6 has no subgroup of order 144, so it is also impossible that $H \cong S_6$. The proof is complete. □

Lemma 3.12 *(a) G is not isomorphic to M_{11} (in this case $H \cong A_6$) i.e. case (13) of Lemma 3.4 does not occur.*

(b) G/N is not isomorphic to A_8 (in this case $H \cong S_6$) i.e. case (3) of Lemma 3.4 does not occur.

(c) G/N is not isomorphic to $U_4(2)$ (in this case $H \cong S_6$ or $H \cong A_6$) i.e. case (15) or case (16) of Lemma 3.4 does not occur.

(d) G is not isomorphic to M_{12} (in this case $H \cong A_6 : 2^2$) i.e. case (12) of Lemma 3.4 does not occur.

Proof By Lemma 3.4 case (13), $G \cong M_{11}$, $H \cong A_6$; case (3), $G/N \cong A_8$, $H \cong S_6$; case (15) or (16), $G/N \cong U_4(2)$, $H \cong S_6$ or $H \cong A_6$. (a), (b), (c) are consequences of Lemma 3.11.

By Lemma 3.4 case (12), $H \cong M_{10} : 2 \cong A_6 : 2^2$. As $|H| = |A_6 : 2^2| = 2^5.3^2.5$ and $H \cap H^a$ has index 5 in H, $|H \cap H^a| = 2^5.3^2 = 288$. Since A_6 has no subgroup of order 72, it follows that $A_6 : 2^2$ has no subgroup of order $72.2^2 = 288$. This contradicts $|H \cap H^a| = 288$, and the proof is complete. □

Lemma 3.13 *Let G and H be as in Theorem 2 with G primitive, and suppose that $H \cong S_5$. Let K be subgroup of H satisfying $K \cong S_4$. Let k be the number of points in Γ fixed by K. Then G has $k-1$ suborbits of length 5.*

Proof Since K is maximal in H, we have, for $\beta \in \mathrm{Fix}_{\Gamma}(K) - \{\alpha\}$, $|\beta^H| = |H : K| = 5$ and by Lemma 2.3 of [9], $\beta^H \cap \mathrm{Fix}_{\Gamma}(K) = \{\beta\}$. So H has $k-1$ orbits of length 5 in Γ. □

Lemma 3.14 *G is not isomorphic to M_{11} and $H \cong S_5$ i.e. case (12) of Lemma3.4 does not occur.*

Proof By Lemma 3.4 (13), $G \cong M_{11}$, $H \cong S_5$. To prove this lemma it suffices to prove that there is no element a such that $G = \langle a, H\rangle$ subject to the relation $|H : H \cap H^a| = 5$. The last relation implies $H \cap H^a \cong S_4$. It is equivalent to show that the action of H on a left coset $\{gH \mid g \in G\}$ has no suborbit of length 5. It suffices by Lemma 3.13 to show that for $K \leq H$ and $K \cong S_4$, K has only one fixed point in $\{gH\}$. We see the sporadic group M_{11} is the automorphism group of a 4-(11,5,1) design and the stabilizer of a block is $H \cong S_5$. Since there is only one conjugacy class of H in G, the action of G on $\{gH\}$ is equivalent to that on the block system $\overline{B} = \{B_i\}$ of the 4-$(11, 5, 1)$ design. Let D be such a design, $X = \{1, 2, \cdots 11\}$ be the point set, and $\overline{B}$ be its set of blocks. Now suppose that the stabilizer of block $B_0 = \{1, 3, 4, 5, 9\}$ is H and $K \leq H$, $K \cong S_4$. Thus K fixes B_0. Thus K induces an action on B_0. Let t be an element of order 3 in K. Then $t = (t_1, t_2, t_3)(t_4, t_5, t_6)(t_7, t_8, t_9)$ and it induces an action on B_0, namely it fixes a sub-block of B_0 of length 3 and fixes every other point B_0, without loss of generality, say $5, 9$. Then $t = (1, 3, 4)(t_4, t_5, t_6)(t_7, t_8, t_9)$. Let χ be the permutation character of degree 66 of M_{11}. Now $\chi = \chi_1 + \chi_2 + \chi_5 + \chi_8$, [6, P18] and elementary calculations lead to $\chi(t) = 3$. This implies that there are just three blocks which are fixed by the action of t in $\overline{B}$. As $\{t_4, t_5, t_6\}, \{t_7, t_8, t_9\}$ are each in two blocks of $\overline{B}$, t determines the three fixed blocks of $\overline{B}$. So the other two blocks fixed by t must be $B_1 = \{t_4, t_5, t_6, 5, 9\}, B_2 = \{t_7, t_8, t_9, 5, 9\}$.

Let u be an element of order 4 in K. Then u fixes B_0 and thus induces an action on B_0. That is, u fixes a sub-block of length 4 as the action of a 4-cycle and fixes the remaining one point. If K fixes another block of $\overline{B}$, then this block must one of B_1 and B_2, without loss of generality, say B_1. Hence u fixes B_1 and induces an action on B_1: u fixes a sub-block of length 4 and fixes a point. Since B_1 must have points 5,9 as above discussed, thus either 5 or 9 must be in the sub-block of length 4, without loss of generality, say 5. It is obvious that the sub-block $\{5^{u^i} | i = 1, 2, 3, 4\}$ of u in B_1 is equal to that of u in B_0. Since their length is 4, $B_0 = B_1$ by the definition of a 4-$(11, 5, 1)$ design and this contradicts that K has another fixed block. This shows that the action of K has only one fixed block and the proof is now complete. □

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