On Hamilton cycles in cubic (m,n)-metacirculant graphs, II

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ABSTRACT

In this paper we continue to investigate the problem of the existence of a Hamilton cycle in connected cubic (m,n)-metacirculant graphs. We show that a connected cubic (m,n)-metacirculant graph $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ has a Hamilton cycle if either $\alpha^2 \equiv 1 \pmod{n}$ or in the case of an odd number μ one of the numbers $(\alpha + 1)$ or $(1 - \alpha + \alpha^2 - \cdots - \alpha^{\mu-2} + \alpha^{\mu-1})$ is relatively prime to n. As a corollary of these results we obtain that every connected cubic (m,n)-metacirculant graph has a Hamilton cycle if m and n are positive integers such that every odd prime divisor of m is not a divisor of $\varphi(n)$ where φ is the Euler φ-function.

1. INTRODUCTION

The problem of the existence of a Hamilton cycle in vertex-transitive graphs has been considered by researchers for many years. Among these graphs, (m,n)-metacirculant graphs introduced in [3] are interesting because the automorphism groups of such graphs contain a transitive subgroup which is a semidirect product of two cyclic groups and so has a rather simple structure. It has been asked [3] whether all connected (m,n)-metacirculant graphs, other than the Petersen graph, have a Hamilton cycle.

For $n = p^t$ with p a prime, connected (m,n)-metacirculant graphs, other than the Petersen graph, have been proved to have a Hamilton cycle [1]. Connected cubic (m,n)-metacirculant graphs, other than the Petersen graph, also have been proved to be hamiltonian for m odd [6], m = 2 [4, 6], and m divisible by 4 [10]. Thus, the remaining values of m, for which we still do not know whether all connected cubic (m,n)-metacirculant graphs have a Hamilton cycle, are of the form $m = 2\mu$ with $\mu \geq 3$ an odd positive integer.

In this paper we continue to investigate the problem of the existence of a Hamilton cycle in connected cubic (m,n)-metacirculant graphs. We will prove here two sufficient conditions for connected cubic (m,n)-metacirculant graphs to be hamiltonian. Namely, we will show that a connected cubic (m,n)-metacirculant graph $G =$

This research was supported in part by the Vietnamese National Basic Research Program in Natural Sciences.

$MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ has a Hamilton cycle if either $\alpha^2 \equiv 1 \pmod{n}$ or in the case of an odd number μ one of the numbers $(\alpha+1)$ or $(1-\alpha+\alpha^2-\dots-\alpha^{\mu-2}+\alpha^{\mu-1})$ is relatively prime to n. As a corollary of these results we obtain that every connected cubic (m,n)-metacirculant graph has a Hamilton cycle if m and n are such that every odd prime divisor of m is not a divisor of φ(n) where φ is the Euler φ-function.

2. PRELIMINARIES

In this paper we consider only finite undirected graphs without loops or multiple edges. If G is a graph, then V(G), E(G) and Aut(G) denote its vertex-set, its edge-set and its automorphism group, respectively. A graph G is called vertex-transitive if the action of Aut(G) on V(G) is transitive. If n is a positive integer, then we write Z_n for the ring of integers modulo n and Z_n^* for the multiplicative group of units in Z_n.

The construction of (m, n)-metacirculant graphs is now described. The reader is referred to [3] for more details and a discussion of the properties of these graphs.

Let m and n be two positive integers, $\alpha \in Z_n^*$, $\mu = \lfloor m/2 \rfloor$ and $S_0, S_1, \dots, S_\mu$ be subsets of Z_n satisfying the following conditions:

(1) $0 \notin S_0 = -S_0$;

(2) $\alpha^m S_r = S_r$ for $0 \le r \le \mu$;

(3) If m is even, then $\alpha^\mu S_\mu = -S_\mu$.

Then we define the *(m,n)-metacirculant graph* $G = MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ to be the graph with vertex-set $V(G) = \{v_j^i : i \in Z_m;\ j \in Z_n\}$ and edge-set $E(G) = \{v_j^i v_h^{i+r} : 0 \le r \le \mu;\ i \in Z_m;\ h, j \in Z_n \text{ and } (h-j) \in \alpha^i S_r\}$, where superscripts and subscripts are always reduced modulo m and modulo n, respectively.

The above construction is designed to allow the permutations ρ with $\rho(v_j^i) = v_{j+1}^i$ and τ with $\tau(v_j^i) = v_{\alpha j}^{i+1}$ to be automorphisms of G. Since the subgroup $< \rho, \tau >$ of Aut(G) generated by ρ and τ is transitive on V(G), (m,n)-metacirculant graphs are vertex-transitive. These graphs were introduced in [3] as a logical generalization of the Petersen graph for the primary reason of providing a class of vertex-transitive graphs in which there might be some new non-hamiltonian connected vertex-transitive graphs. Among these graphs, cubic (m,n)-metacirculant graphs are especially attractive, being at the same time the simplest nontrivial (m,n)-metacirculant graphs and those most likely to be non-hamiltonian because of their small number of edges.

Now we recall a method for lifting a Hamilton cycle in a quotient graph $\overline{G}$ of a graph G to a Hamilton cycle in G. This method will be used in the next section to prove Theorem 4. A permutation β is said to be semiregular if all cycles in the disjoint cycle decomposition of β have the same length. If a graph G has a semiregular automorphism β, then the quotient graph G/β with respect to β is defined as follows [2]. The vertices of G/β are the orbits of the subgroup $< \beta >$ generated by β and two such vertices are adjacent if and only if there is an edge in G joining a vertex of one corresponding orbit to a vertex in the other orbit.

Let β be of order t and $G^0, G^1, \ldots, G^h$ be the subgraphs induced by G on the orbits of $< \beta >$. Let $v_0^i, v_1^i, \ldots, v_{t-1}^i$ be a cyclic labelling of the vertices of G^i under the action of β and $C = G^0 G^i G^j \ldots G^r G^0$ be a cycle of G/β. Consider a path P of G arising from a lifting of C, namely, start at v_0^0 and choose an edge from v_0^0 to a vertex v_a^i of G^i. Then take an edge from v_a^i to a vertex v_b^j of G^j following G^i in C. Continue in this way until returning to a vertex v_d^0 of G^0. The set of all paths that can be constructed in this way using C is called in [2] the *coil* of C and is denoted by coil(C).

We will use in the next section the following results proved in [7], [8] and [9].

Lemma 1 [7]. *Let t be the order of a semiregular automorphism β of a graph G and G^0 be the subgraph induced by G on an orbit of $< \beta >$. If there exists a Hamilton cycle C in G/β such that coil(C) contains a path P whose terminal vertices are distance d apart in the G^0 where P starts and terminates and gcd(d,t) = 1, then G has a Hamilton cycle.*

Lemma 2 [8]. *Let $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ be a cubic (m,n)-metacirculant graph such that $m > 2$ is even, $S_0 = \emptyset, S_i = \{s\}$ with $0 \leq s < n$ for some $i \in \{1, 2, \ldots, \mu - 1\}$, $S_j = \emptyset$ for all $i \neq j \in \{1, 2, \ldots, \mu - 1\}$ and $S_\mu = \{k\}$ with $0 \leq k < n$. Then*

(i) If G is connected, then either i is odd and gcd(i,m) = 1 or i is even, μ is odd and gcd(i,m) = 2.

(ii) If i is odd and gcd(i,m) = 1, then G is isomorphic to the cubic (m,n)-metacirculant graph $G' = MC(m, n, \alpha', S'_0, S'_1, \ldots, S'_\mu)$ with $\alpha' = \alpha^i, S'_0 = \emptyset, S'_1 = \{s\}, S'_2 = \cdots = S'_{\mu-1} = \emptyset$ and $S'_\mu = \{k\}$.

(iii) If i is even, μ is odd, gcd(i,m) = 2 and $i = 2^r i'$ with $r \geq 1$ and i' odd, then G is isomorphic to the cubic (m,n)-metacirculant graph $G'' = MC(m, n, \alpha'', S''_0, S''_1, \ldots, S''_\mu)$ with $\alpha'' = \alpha^{i'}, S''_0 = S''_1 = \cdots = S''_{2^r-1} = \emptyset, S''_{2^r} = \{s\}, S''_{2^r+1} = \cdots = S''_{\mu-1} = \emptyset$ and $S''_\mu = \{k\}$.

Lemma 3 [8]. *(i) Let* $G = MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ *be a cubic (m,n)-metacirculant graph such that* $m > 2$ *is even,* $S_0 = \emptyset, S_1 = \{s\}, S_2 = \dots = S_{\mu-1} = \emptyset$ *and* $S_\mu = \{k\}$. *Then G is connected if and only if gcd(p,n) = 1, where p is* $[k - s(1 + \alpha + \alpha^2 + \dots + \alpha^{\mu-1})]$ *reduced modulo n.*

(ii) Let $G = MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ *be a cubic (m,n)-metacirculant graph such that* $m > 2$ *is even,* $\mu = \lfloor m/2 \rfloor$ *is odd,* $S_0 = S_1 = \dots = S_{2^r-1} = \emptyset$ *with* $r \geq 1, S_{2^r} = \{s\}, S_{2^r+1} = \dots = S_{\mu-1} = \emptyset$ *and* $S_\mu = \{k\}$. *Then G is connected if and only if gcd(q,n) = 1, where q is* $[k(1+\alpha+\alpha^2+\dots+\alpha^{2^r-1}) - s(1+\alpha+\alpha^2+\dots+\alpha^{\mu-1})]$ *reduced modulo n.*

Lemma 4 [9]. *Let* $G = MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ *be a connected cubic (m,n)-metacirculant graph such that m is even, greater than 2 and not divisible by 4,* $S_0 = S_1 = \dots = S_{2^r-1} = \emptyset$ *with* $r \geq 1, S_{2^r} = \{s\}$ *with* $0 \leq s < n, S_{2^r+1} = \dots = S_{\mu-1} = \emptyset$ *and* $S_\mu = \{k\}$ *with* $0 \leq k < n$. *Let* $a = gcd(\alpha - 1, n)$ *and* $b = gcd(1 - \alpha + \alpha^2 - \dots + \alpha^{\mu-1}, n)$. *Then G has a Hamilton cycle if any one of the following conditions is met:*

(i) Either $gcd(n/(ab), \mu a - 1) = 1$; *or*

(ii) $b = 1$.

Now we recall the definition of a brick product of a cycle with a path defined in [4]. This product plays a role in the proof of Theorem 1 in the next section. Let C_n with $n \geq 3$ and P_m with $m \geq 1$ be the graphs with vertex-sets $V(C_n) = \{u_1, u_2, \dots, u_n\}, V(P_m) = \{v_1, v_2, \dots, v_{m+1}\}$ and edge-sets $E(C_n) = \{u_1u_2, u_2u_3, \dots, u_nu_1\}, E(P_m) = \{v_1v_2, v_2v_3, \dots, v_mv_{m+1}\}$, respectively. The *brick product* $C_n^{[m+1]}$ of C_n with P_m is defined as follows [4]. The vertex-set of $C_n^{[m+1]}$ is the cartesian product $V(C_n) \times V(P_m)$. The edge-set of $C_n^{[m+1]}$ consists of all pairs of the form $(u_i, v_h)(u_{i+1}, v_h)$ and $(u_1, v_h)(u_n, v_h)$, where i = 1, 2, ..., n−1 and h = 1, 2, ..., m + 1, together with all pairs of the form $(u_i, v_h)(u_i, v_{h+1})$, where $i + h \equiv 0 \pmod 2$, i = 1, 2, ..., n and h = 1, 2, ..., m.

The following result has been proved in [4].

Lemma 5 [4]. *Consider the brick product* $C_n^{[m]}$ *with n even. Let* $C_{n,1}$ *and* $C_{n,m}$ *denote the two cycles in* $C_n^{[m]}$ *on the vertex-sets* $\{(u_i, v_1) : i = 1, 2, \dots, n\}$ *and* $\{(u_i, v_m) : i = 1, 2, \dots, n\}$, *respectively. Let F denote an arbitrary perfect matching joining the vertices of degree 2 in* $C_{n,1}$ *with the vertices of degree 2 in* $C_{n,m}$. *If X is a graph obtained by adding the edges of F to* $C_n^{[m]}$, *then X has a Hamilton cycle.*

3. MAIN RESULTS

In this section we will prove two sufficient conditions for connected cubic (m,n)-metacirculant graphs to be hamiltonian. These conditions will be expected helpful in further investigation of the problem of the existence of a Hamilton cycle in connected cubic (m,n)-metacirculant graphs. As a corollary of one of these conditions we obtain that every connected cubic (m,n)-metacirculant graph has a Hamilton cycle if m and n are positive integers such that every odd prime divisor of m is not a divisor of $\varphi(n)$ where φ is the Euler φ-function. This is a partial solution of the above mentioned problem.

Theorem 1. *Let $G = MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ be a connected cubic (m,n)-metacirculant graph such that $\alpha^2 \equiv 1 \pmod{n}$. Then G possesses a Hamilton cycle.*

Proof. Let $G = MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ be a connected cubic (m,n)-metacirculant graph such that $\alpha^2 \equiv 1 \pmod{n}$. Suppose that G is isomorphic to the Petersen graph. Then mn = 10 because the orders of G and the Petersen graph are equal to mn and 10, respectively. Hence m is equal to one of the numbers 1, 2, 5 or 10. If m = 1, then by definition G is a circulant graph. So G is a Cayley graph. If m = 5 or 10, then n = 2 or 1, respectively. Therefore, $\alpha = 1$. By [3, Theorem 9], G is a Cayley graph. If m = 2, then the hypothesis $\alpha^2 \equiv 1 \pmod{n}$ implies by [3, Theorem 9] again that G is also a Cayley graph. Thus, in all cases G is Cayley. This contradicts the well-known fact that the Petersen graph is not a Cayley graph. It follows that G cannot be isomorphic to the Petersen graph.

If m is odd or m = 2 or m is divisible by 4, then by the results obtained in [4, 6, 10] G has a Hamilton cycle. If $S_0 \neq \emptyset$, then by [6] G also possesses a Hamilton cycle. Therefore, we may assume from now on that *m is even, greater than 2 and not divisible by 4 and $S_0 = \emptyset$*. Since G is a cubic (m,n)-metacirculant graph, this implies that only the following may happen:

(i) $S_0 = \emptyset, S_i = \{s\}$ with $0 \leq s < n$ for some $i \in \{1, 2, \dots, \mu - 1\}, S_j = \emptyset$ for all $i \neq j \in \{1, 2, \dots, \mu - 1\}$ and $S_\mu = \{k\}$ with $0 \leq k < n$;

(ii) $S_0 = \dots = S_{\mu-1} = \emptyset$ and $|S_\mu| = 3$.

Since G is connected and $m > 2$ is even, (ii) cannot occur. So only (i) may happen. By Lemma 2, without loss of generality we may assume that $G = MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ has one of the following forms:

1. $S_0 = \emptyset, S_1 = \{s\}, S_2 = \dots = S_{\mu-1} = \emptyset$ and $S_\mu = \{k\}$;

2. $S_0 = \dots = S_{2^r-1} = \emptyset$ with $r \geq 1, S_{2^r} = \{s\}, S_{2^r+1} = \dots = S_{\mu-1} = \emptyset$ and $S_\mu = \{k\}$.

We consider these possibilities in turn. Below we will use the hypothesis $\alpha^2 \equiv 1 \pmod{n}$ frequently without mention. So the reader should keep it in mind.

Case 1: $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ with $S_0 = \emptyset, S_1 = \{s\}, S_2 = \cdots = S_{\mu-1} = \emptyset$ and $S_\mu = \{k\}$.

An edge in G of the type $v_j^i v_{j+\alpha^i s}^{i+1}$ is called an S_1-edge, and of the type $v_j^i v_{j+\alpha^i k}^{i+\mu}$ an S_μ-edge. A cycle C in G is called an S_1-cycle if every edge in C is an S_1-edge. Consider S_1-cycles in G. Since every vertex in G is incident with just two S_1-edges, it must be contained in exactly one S_1-cycle. So two S_1-cycles either coincide or are disjoint. Further, it is clear that any S_1-cycle P_j in G must contain a vertex v_y^0 for some $y \in Z_n$ and therefore can be represented in the form

$$P_j = P(v_y^0)P(v_{y+z}^0)P(v_{y+2z}^0)\ldots,$$

where z is $\mu s + \mu\alpha s$ and

$$P(v_h^0) = v_h^0 v_{h+s}^1 v_{h+s+\alpha s}^2 v_{h+2s+\alpha s}^3 v_{h+2s+2\alpha s}^4 \cdots v_{h+(\mu-1)s+(\mu-1)\alpha s}^{2\mu-2} v_{h+\mu s+(\mu-1)\alpha s}^{2\mu-1}.$$

It follows that two vertices v_f^i and v_g^{i+2} of G are vertices at distance 2 apart in the same S_1-cycle P_j if and only if $g = f + s + \alpha s$ in Z_n. It is also not difficult to see that all S_1-cycles in G are isomorphic to each other and have an even length ℓ.

If G has only one S_1-cycle, then this cycle is trivially a Hamilton cycle of G. Therefore, we assume that G has at least two distinct S_1-cycles. Let v_f^i and v_g^{i+2} with i even be two vertices at distance 2 apart in the same S_1-cycle P_j. Then the vertices of G adjacent to v_f^i and v_g^{i+2} by S_μ-edges are $v_{f'}^{i+\mu}$ and $v_{g'}^{i+2+\mu}$, respectively, where $f' = f + \alpha^i k = f + k$ and $g' = g + \alpha^{i+2}k = g + k$. Since $g = f + s + \alpha s$ in Z_n, we have $g' = g + k = f + s + \alpha s + k = f' + s + \alpha s$ in Z_n. Thus $v_{f'}^{i+\mu}$ and $v_{g'}^{i+2+\mu}$ are vertices at distance 2 apart in the same S_1-cycle $P_{j'}$. Moreover, since μ is odd, the superscripts $i + \mu$ and $i + 2 + \mu$ of respectively $v_{f'}^{i+\mu}$ and $v_{g'}^{i+2+\mu}$ are odd.

Let $C_\ell^{[r]}$ be the brick product of a cycle C_ℓ with a path P_{r-1}, where C_ℓ is isomorphic to S_1-cycles of G and r is the number of distinct S_1-cycles in G. Denote by $C_{\ell,1}$ and $C_{\ell,r}$ the two cycles in $C_\ell^{[r]}$ on the vertex-sets $\{(u_i, v_1) : i = 1, 2, \ldots, \ell\}$ and $\{(u_i, v_r) : i = 1, 2, \ldots, \ell\}$, respectively. Using the property of G proved in the preceding paragraph and the fact that G is a connected cubic graph, it is not difficult to see that G is isomorphic to a graph X obtained from $C_\ell^{[r]}$ by adding the edges of a perfect matching joining the vertices of degree 2 in $C_{\ell,1}$ with the vertices of degree 2 in $C_{\ell,r}$. By Lemma 5, X has a Hamilton cycle. Therefore, G has a Hamilton cycle in Case 1.

Case 2:. $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ with $S_0 = \cdots = S_{2^r-1} = \emptyset$ for some $r \geq 1, S_{2^r} = \{s\}, S_{2^r+1} = \cdots = S_{\mu-1} = \emptyset$ and $S_\mu = \{k\}$.

An edge in G of the type $v_j^i v_{j+\alpha^i s}^{i+2^r}$ is called an S_{2^r}-edge, and of the type $v_j^i v_{j+\alpha^i k}^{i+\mu}$ an S_μ-edge. A walk W in G is called an S_{2^r}-walk if every edge in W is an S_{2^r}-edge. Since an S_{2^r}-edge connects vertices with superscripts of the same parity, either all superscripts of vertices of an S_{2^r}-walk are even or they are all odd modulo m. In the former case, an S_{2^r}-walk is called of type A and in the latter case, it is called of type B.

Since G is connected, by Lemma 3,

$$\begin{aligned} &gcd([k(1+\alpha+\cdots+\alpha^{2^r-1})-s(1+\alpha+\cdots+\alpha^{\mu-1})],n)= \\ &gcd([k(\alpha+1)(1+\alpha^2+\alpha^4+\cdots+\alpha^{2^r-2})-s(1+\alpha+ \\ &\quad \alpha^2+\cdots+\alpha^{\mu-1})],n)=1. \end{aligned} \tag{3.1}$$

By the definition of (m,n)-metacirculant graphs, we have $\alpha^\mu k \equiv -k \pmod n \iff (\alpha^\mu+1)k \equiv 0 \pmod n$. Therefore, since $\alpha^2 \equiv 1 \pmod n$ and μ is odd,

$$k(\alpha+1) \equiv k(\alpha^\mu+1) \equiv 0 \pmod n. \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$gcd(s,n)=1, \text{ and} \tag{3.3}$$

$$gcd(1+\alpha+\alpha^2+\cdots+\alpha^{\mu-1},n)=1. \tag{3.4}$$

On the other hand, by $\alpha^2 \equiv 1 \pmod n$, we have

$$\begin{aligned} \mu \equiv\, &1+\alpha^2+\cdots+\alpha^{2(\mu-1)} \equiv (1-\alpha+\alpha^2-\cdots \\ &-\alpha^{\mu-2}+\alpha^{\mu-1})(1+\alpha+\cdots+\alpha^{\mu-1}) \pmod n. \end{aligned} \tag{3.5}$$

Let $b = gcd(1-\alpha+\alpha^2-\cdots+\alpha^{\mu-1},n)$. Then by (3.4) and (3.5)

$$gcd(\mu,n)=b. \tag{3.6}$$

This implies in particular that b is odd because μ is odd. Since $\alpha \in Z_n^*$, we also have

$$gcd(\alpha,n)=1. \tag{3.7}$$

Since $\alpha^2 \equiv 1 \pmod n$, $(\alpha+1)(\alpha-1) \equiv 0 \pmod n$. On the other hand, $gcd(1-\alpha+\alpha^2-\cdots+\alpha^{\mu-1},\alpha-1,n)=1$ because of (3.7). Therefore, $b = gcd(1-\alpha+\alpha^2-\cdots+\alpha^{\mu-1},n)$ is a divisor of $gcd(\alpha+1,n)$. Thus, $b = gcd(1-\alpha+\alpha^2-\cdots+\alpha^{\mu-1},n) = gcd(\mu,n)$ is odd, and $\alpha+1 = b^u x$ with $u \geq 1$.

Let $G' = MC(m,n,\alpha',S_0',S_1',\ldots,S_\mu')$ be a cubic (m,n)-metacirculant graph such that $\alpha' = \alpha$, $S_{2^r}' = \{1\}$, $S_\mu' = \{0\}$ and $S_j' = \emptyset$ for all $j \neq 2^r$ and μ. Further, let $V(G') = \{w_j^i : i \in Z_m;\ j \in Z_n\}$. Since gcd(s,n) = 1 by (3.3), it is not difficult to verify that the mapping

$$\psi : V(G') \to V(G) : \begin{cases} w_j^i \mapsto v_{js}^i & \text{if i is even,} \\ w_j^i \mapsto v_{js+k}^i & \text{if i is odd} \end{cases}$$

is an isomorphism of G' and G. Therefore, without loss of generality we may assume that G is a cubic (m,n)-metacirculant graph $MC(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ such that

$$b = gcd(1 - \alpha + \alpha^2 - \dots + \alpha^{\mu-1}, n) = gcd(\mu, n) \text{ is odd,} \tag{3.8}$$

$$\alpha + 1 = b^u x \text{ with } u \geq 1, \tag{3.9}$$

$$S_{2^r} = \{1\},\ S_\mu = \{0\} \text{ and } S_j = \emptyset \text{ for all } j \neq 2^r \text{ and } \mu.$$

Now we prove the following claim which is needed to determine when two vertices v_j^i and v_f^i of G belong to the same S_{2^r}-cycle.

Claim 1. *Two vertices v_j^i and v_f^i of G belong to the same S_{2^r}-cycle if and only if $f \equiv j \pmod b$.*

Proof. Since every vertex of G is incident with just two S_{2^r}-edges, the S_{2^r}-cycle Q containing v_j^i can be represented in the form

$$Q = Q(v_j^i)Q(v_{j+z}^i)Q(v_{j+2z}^i) \dots, \tag{3.10}$$

where $z \equiv \alpha^i + \alpha^{i+2^r} + \alpha^{i+2\cdot 2^r} + \dots + \alpha^{i+(\mu-2)2^r} + \alpha^{i+(\mu-1)2^r} \equiv \mu\alpha^i \pmod n$, and

$$Q(v_h^i) = v_h^i v_{(h+\alpha^i)}^{i+2^r} v_{(h+\alpha^i+\alpha^{i+2^r})}^{i+2\cdot 2^r} \dots v_{(h+z-\alpha^{i+(\mu-1)2^r})}^{i+(\mu-1)2^r}$$
$$= v_h^i v_{h+\alpha^i}^{i+2^r} v_{h+2\alpha^i}^{i+2\cdot 2^r} \dots v_{h+(\mu-1)\alpha^i}^{i+(\mu-1)2^r}.$$

Thus, the vertices of Q with superscript i are v_j^i, v_{j+z}^i, v_{j+2z}^i, $\cdots$ because i, $i + 2^r$, $i + 2\cdot 2^r$, $\cdots$, $i + (\mu - 1)2^r$are distinct from each other modulo m. It follows that v_f^i belongs to Q if and only if $f \equiv j + tz \pmod n$ for some integer t.

Since (3.7) holds,

$$gcd(z, n) = gcd(\mu, n) = b. \tag{3.11}$$

Therefore, if $f \equiv j + tz \pmod n$, then (3.8) and (3.11) imply that $f \equiv j \pmod b$. Conversely, if $f \equiv j \pmod b$, then $f = j + u_1 b$ for some integer u_1. Since (3.11) holds, there exist integers u_2 and u_3 such that $b = u_2 z + u_3 n$. So $f = j + u_1 u_2 z + u_1 u_3 n$. This means that $f \equiv j + tz \pmod n$ for some integer t. Thus, v_f^i belongs to Q if and only if $f \equiv j \pmod b$. □.

Consider S_{2^r}-cycles in G. Since every vertex of G is incident with just two S_{2^r}-edges, it must be contained in exactly one S_{2^r}-cycle. So any two S_{2^r}-cycles either coincide or are disjoint. Further, since μ is odd, the numbers $0, 2^r, 2\cdot 2^r, 3\cdot 2^r, \cdots, (\mu-1)2^r$ are all even numbers modulo m. Hence every S_{2^r}-cycle Q of type A must contain a vertex v_j^0 and every S_{2^r}-cycle Q of type B must contain a vertex v_j^μ for some $j \in Z_n$ because Q can be represented in the form (3.10). Hence, by Claim 1, the S_{2^r}-cycles $A^0, A^1, A^2, \ldots, A^{b-2}, A^{b-1}, B^0, B^1, B^2, \ldots, B^{b-2}$ and B^{b-1} containing $v_0^0, v_{b-1}^0, v_{b-2}^0, \cdots, v_2^0, v_1^0, v_0^\mu, v_1^\mu, v_2^\mu, \ldots, v_{b-2}^\mu$ and v_{b-1}^μ, respectively, are all disjoint S_{2^r}-cycles of G. So each vertex of G must be contained in exactly one of these S_{2^r}-cycles. The cycles $A^0, A^1, A^2, \ldots, A^{b-1}$ are of type A and the cycles $B^0, B^1, B^2, \ldots, B^{b-1}$ are of type B. We also note that each edge of each $A^\ell, \ell = 0, 1, \ldots, b-1$, has the form $v_j^i v_{j+1}^{i+2^r}$ with i even, whereas each edge of each $B^\ell, \ell = 0, 1, \ldots, b-1$, has the form $v_j^i v_{j+\alpha}^{i+2^r}$ with i odd.

Claim 1 is very useful in determining which S_{2^r}-cycle A^ℓ or B^ℓ a given vertex belongs to. For example, to determine which S_{2^r}-cycles A^ℓ or B^ℓ the vertices $v_\alpha^{(b-3)2^r}$ and $v_{2+\alpha}^{(b-1)2^r+\mu}$ belong to, we note that $v_\alpha^{(b-3)2^r}$ and $v_{2+\alpha}^{(b-1)2^r+\mu}$ are contained in the S_{2^r}-paths $v_\alpha^{(b-3)2^r} v_{\alpha-1}^{(b-4)2^r} v_{\alpha-2}^{(b-5)2^r} \cdots v_{\alpha-(b-3)}^0$ and $v_{2+\alpha}^{(b-1)2^r+\mu} v_2^{(b-2)2^r+\mu} v_{2-\alpha}^{(b-3)2^r+\mu} \cdots v_{(2+\alpha)-(b-1)\alpha}^\mu$, respectively. Since (3.9) holds, $\alpha + 1 \equiv 0 \pmod b$ and $(-\alpha) \equiv 1 \pmod b$. So $\alpha - (b-3) = (\alpha+1) - b + 2 \equiv 2 \pmod b$ and $(2+\alpha) - (b-1)\alpha = 1 + (1+\alpha) + (b-1)(-\alpha) \equiv 1 + (b-1) \equiv 0 \pmod b$. By Claim 1, $v_{\alpha-(b-3)}^0$ is contained in the S_{2^r}-cycle containing v_2^0, i.e., A^{b-2} and $v_{(2+\alpha)-(b-1)\alpha}^\mu$ is contained in the S_{2^r}-cycle containing v_0^μ, i.e., B^0. Therefore, $v_\alpha^{(b-3)2^r}$ and $v_{2+\alpha}^{(b-1)2^r+\mu}$ are contained in A^{b-2} and B^0, respectively. Similar applications of Claim 1 will be used frequently without mention.

We introduce now the following definition similar to that of Bannai's work [5]. An alternating cycle C of G is defined to be a cycle the sequence of adjacent edges of which are $e_1, f_1, e_2, f_2, \ldots, e_{2t}, f_{2t}$, where $e_i, i = 1, 2, \ldots, 2t$, are S_{2^r}-edges and $f_i, i = 1, 2, \ldots, 2t$, are S_μ-edges. For convenience, we will consider an alternating cycle C as a sequence of adjacent edges and will simply write $C = e_1 f_1 e_2 f_2 \ldots e_{2t} f_{2t}$.

For any vertex v_j^i of G, we have the following alternating cycle $AC(v_j^i) = e_1(v_j^i) f_1(v_j^i) e_2(v_j^i) f_2(v_j^i) e_3(v_j^i) f_3(v_j^i) e_4(v_j^i) f_4(v_j^i)$, where

$$e_1(v_j^i) = v_j^i v_{j+\alpha^i}^{i+2^r},$$

$$f_1(v_j^i) = v_{j+\alpha^i}^{i+2^r} v_{j+\alpha^i}^{i+2^r+\mu},$$

$$e_2(v_j^i) = v_{(j+\alpha^i)}^{(i+2^r+\mu)} v_{(j+\alpha^i+\alpha^{i+2^r+\mu})}^{(i+2\cdot 2^r+\mu)}$$

$$= v_{(j+\alpha^i)}^{(i+2^r+\mu)} v_{(j+\alpha^i(1+\alpha))}^{(i+2\cdot 2^r+\mu)},$$

$$f_2(v_j) = v_{(j+\alpha^i(1+\alpha))} v_{(j+\alpha^i(1+\alpha))},$$

$$e_3(v_j^i) = v_{(j+\alpha^i(1+\alpha))}^{(i+2\cdot 2^r)} v_{(j+\alpha^i(1+\alpha)-\alpha^{i+2^r})}^{(i+2^r)}$$

$$= v_{(j+\alpha^i(1+\alpha))}^{(i+2\cdot 2^r)} v_{(j+\alpha^{i+1})}^{(i+2^r)},$$

$$f_3(v_j^i) = v_{(j+\alpha^{i+1})}^{(i+2^r)} v_{(j+\alpha^{i+1})}^{(i+2^r+\mu)},$$

$$e_4(v_j^i) = v_{(j+\alpha^{i+1})}^{(i+2^r+\mu)} v_{(j+\alpha^{i+1}-\alpha^{i+\mu})}^{(i+\mu)}$$

$$= v_{(j+\alpha^{i+1})}^{(i+2^r+\mu)} v_j^{i+\mu},$$

$$f_4(v_j^i) = v_j^{i+\mu} v_j^i.$$

For simplicity of notation we will write $e_1, f_1, \dots, e_4, f_4$ instead of $e_1(v_j^i), f_1(v_j^i), \dots, e_4(v_j^i), f_4(v_j^i)$, respectively. In the context it will be clear which vertex v_j^i we deal with. An alternating cycle $AC(v_j^i)$ plays an important role in the proof of Theorem 1 in Case 2.

A construction of a Hamilton cycle in G in Case 2 will be based on the following property of $AC(v_j^i)$.

Claim 2. *If $b \geq 3$, then for any vertex v_j^i of G, the edges e_1, e_2, e_3 and e_4 of the alternating cycle $AC(v_j^i) = e_1 f_1 e_2 f_2 e_3 f_3 e_4 f_4$ belong to distinct S_{2^r}-cycles.*

Proof. If e_1 is an edge of an S_{2^r}-cycle of type A (resp. type B), then e_3 is also an edge of an S_{2^r}-cycle of type A (resp. type B) and e_2 and e_4 are edges of S_{2^r}-cycles of type B (resp. type A). This is clear from the definition of an alternating cycle $AC(v_j^i)$. Since any S_{2^r}-cycle of type A and any S_{2^r}-cycle of type B are disjoint, to prove Claim 2, it is sufficient to show that the S_{2^r}-cycle containing e_1 is different from the S_{2^r}-cycle containing e_3 and the S_{2^r}-cycle containing e_2 is different from the S_{2^r}-cycle containing e_4.

Suppose that e_1 and e_3 are edges of the same S_{2^r}-cycle Q. Then $v_{j+\alpha^i}^{i+2^r}$ and $v_{j+\alpha^{i+1}}^{i+2^r}$ are vertices of Q. By Claim 1, $j+\alpha^{i+1} \equiv j+\alpha^i \pmod b \iff \alpha^i(\alpha-1) \equiv 0 \pmod b$. This implies by (3.7) and (3.8) that $\alpha - 1 \equiv 0 \pmod b \iff \alpha + 1 \equiv 2 \pmod b$ which is impossible because $b \geq 3$ and $\alpha + 1 = b^u x$ with $u \geq 1$ by (3.9). The obtained contradiction shows that the S_{2^r}-cycle containing e_1 is different from the S_{2^r}-cycle containing e_3. Similarly, we can prove that the S_{2^r}-cycle containing e_2 is different from the S_{2^r}-cycle containing e_4. □

Now we consider separately three subcases.

Subcase 2.1: b = 1. In this subcase, G has a Hamilton cycle by Lemma 4(ii).

Subcase 2.2: b = 3. First assume that the vertices $v_0^\mu, v_{3\alpha}^{3\cdot 2^r+\mu}$ and $v_3^{3\cdot 2^r+\mu}$ of B^0 are pairwise distinct (Fig. 1). This implies that the vertices $v_\alpha^{2^r}, v_{4\alpha}^{4\cdot 2^r}$ and $v_{\alpha+3}^{4\cdot 2^r}$ of A^2 are also pairwise distinct. Further, the edge $v_{4\alpha}^{4\cdot 2^r+\mu} v_{5\alpha}^{5\cdot 2^r+\mu}$ is an edge of the subpath P of B^0 not containing v_0^μ and connecting $v_\alpha^{2^r+\mu}$ with $v_3^{3\cdot 2^r+\mu}$. Moreover, $v_{4\alpha}^{4\cdot 2^r+\mu}$ and $v_{5\alpha}^{5\cdot 2^r+\mu}$ are not the endvertices of P. Such a graph G possesses a Hamilton cycle shown in Figure 1.

Next assume that $v_{3\alpha}^{3\cdot 2^r+\mu} = v_3^{3\cdot 2^r+\mu}$ but $v_{3\alpha}^{3\cdot 2^r+\mu} \neq v_0^\mu$ (Fig. 2). If $v_0^\mu \neq v_6^{6\cdot 2^r+\mu}$, then since $3\alpha \equiv 3 \pmod{n}, 4\alpha = 3\alpha + \alpha \equiv 3 + \alpha \pmod{n}$ and $4\alpha + 1 \equiv 4 + \alpha \pmod{n}$. Therefore, $v_{4\alpha}^{4\cdot 2^r+\mu} = v_{3+\alpha}^{4\cdot 2^r+\mu}$ and $v_{4\alpha+1}^{5\cdot 2^r+\mu} = v_{4+\alpha}^{5\cdot 2^r+\mu}$. Further, the edge $v_{4\alpha}^{4\cdot 2^r+\mu} v_{5\alpha}^{5\cdot 2^r+\mu}$ is an edge of the subpath P of B^0 not containing v_0^μ and connecting $v_\alpha^{2^r+\mu}$ with $v_6^{6\cdot 2^r+\mu} = v_{6\alpha}^{6\cdot 2^r+\mu}$. Moreover, $v_{4\alpha}^{4\cdot 2^r+\mu}$ and $v_{5\alpha}^{5\cdot 2^r+\mu}$ are not the endvertices of P. Such a graph G possesses a Hamilton cycle shown in Figure 2. If $v_0^\mu = v_6^{6\cdot 2^r+\mu}$, then $6\cdot 2^r + \mu \equiv \mu \pmod{m}$ and $6 \equiv 0 \pmod{n}$. So $\mu = 3$ and n = 3 or 6. Therefore, $v_{3\alpha}^{3\cdot 2^r+\mu} = v_{3\alpha}^3 \neq v_0^3$. This implies that $3\alpha \not\equiv 0 \pmod{n} \iff 3 \not\equiv 0 \pmod{n}$. So $n \neq 3$. Thus, this possibility happens only if $\mu = 3$ and n = 6. We leave to the reader to verify that for these values of μ and n the graph G also has a Hamilton cycle.

Finally assume that $v_0^\mu = v_{3\alpha}^{3\cdot 2^r+\mu} = v_3^{3\cdot 2^r+\mu}$. From $v_0^\mu = v_3^{3\cdot 2^r+\mu}$ it follows that $3\cdot 2^r + \mu \equiv \mu \pmod{m}$ and $3 \equiv 0 \pmod{n}$. So $\mu = 3$ and n = 3. We again leave to the reader to verify that for these values of μ and n the graph G also has a Hamilton cycle. This completes the proof for Subcase 2.2.

Subcase 2.3: $b \geq 5$. Let e be an S_{2^r}-edge and C be the S_{2^r}-cycle containing e. From C by deleting the edge e we obtain a path which is called the S_{2^r}-complementing path of e and is denoted by CP(e). Let $AC(v_j^i) = e_1 f_1 e_2 f_2 e_3 f_3 e_4 f_4$ be the alternating cycle for v_j^i defined earlier. From $AC(v_j^i)$ by deleting the edge e_1 we obtain a path which is called the alternating path for v_j^i and is denoted by $AP(v_j^i)$, i.e., $AP(v_j^i) = f_1 e_2 f_2 e_3 f_3 e_4 f_4$. In its turn, from $AP(v_j^i)$ by replacing each $e_i, i = 2, 3, 4$, by its S_{2^r}-complementing path $CP(e_i)$ we can get another path in G which we denote by $\overline{AP}(v_j^i)$.

The idea for a construction of a Hamilton cycle of G in this subcase is as follows. Let a cycle C in G containing all vertices of some S_{2^r}-cycles and only these vertices have been constructed. We choose an appropriate vertex v_j^i of C such that the S_{2^r}-edge $v_j^i v_{j+\alpha^i}^{i+2^r}$ is an edge of C and the vertices v_j^i and $v_{j+\alpha^i}^{i+2^r}$ are the only common vertices of C and $\overline{AP}(v_j^i)$. Then by replacing the edge $v_j^i v_{j+\alpha^i}^{i+2^r}$ by $\overline{AP}(v_j^i)$ we get from C a longer cycle C' containing all vertices of a larger number of S_{2^r}-cycles and only these vertices. By appropriate choices of vertices v_j^i we can continue this

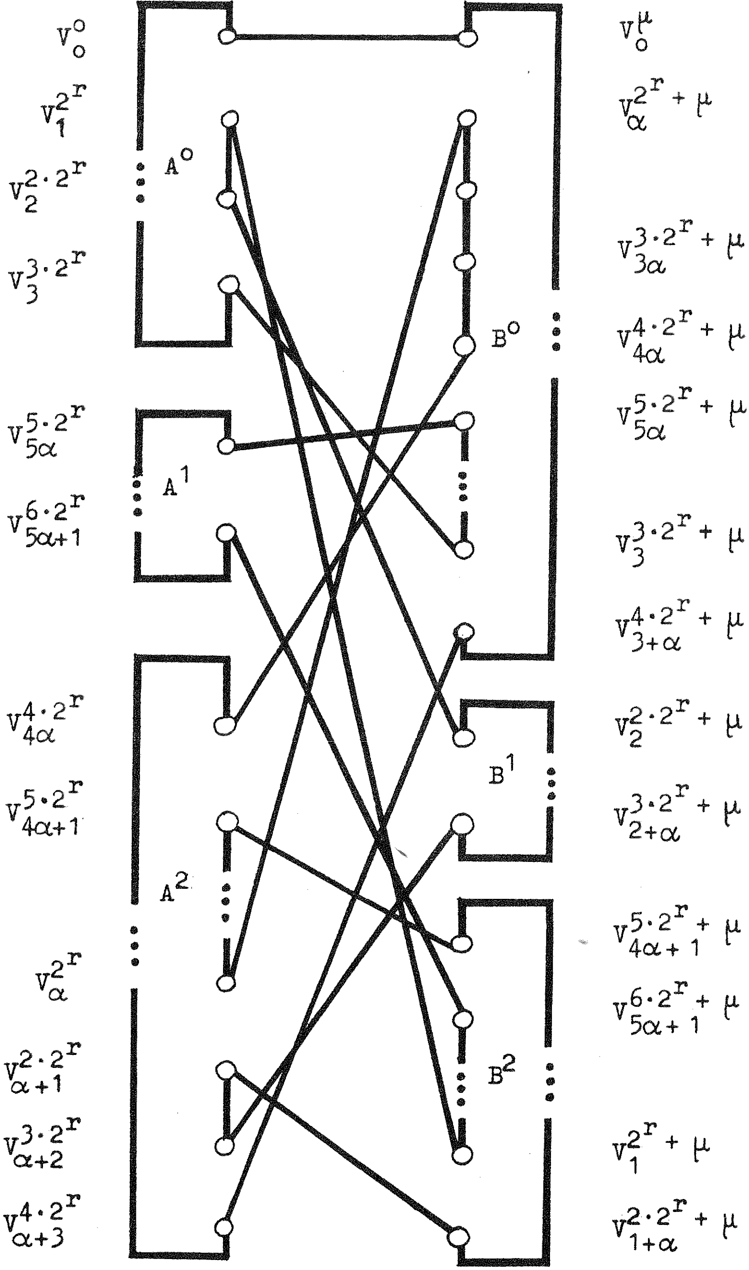

V_o^o
V_o^μ
$V_1^{2^r}$
$V_\alpha^{2^r}+\mu$
A^o
$V_2^{2\cdot 2^r}$
$V_3^{3\cdot 2^r}$
$V_{3\alpha}^{3\cdot 2^r}+\mu$
B^o
$V_{4\alpha}^{4\cdot 2^r}+\mu$
$V_{5\alpha}^{5\cdot 2^r}$
$V_{5\alpha}^{5\cdot 2^r}+\mu$
A^1
$V_{5\alpha+1}^{6\cdot 2^r}$
$V_3^{3\cdot 2^r}+\mu$
$V_{3+\alpha}^{4\cdot 2^r}+\mu$
$V_{4\alpha}^{4\cdot 2^r}$
$V_2^{2\cdot 2^r}+\mu$
B^1
$V_{4\alpha+1}^{5\cdot 2^r}$
$V_{2+\alpha}^{3\cdot 2^r}+\mu$
A^2
$V_{4\alpha+1}^{5\cdot 2^r}+\mu$
$V_\alpha^{2^r}$
$V_{5\alpha+1}^{6\cdot 2^r}+\mu$
$V_{\alpha+1}^{2\cdot 2^r}$
B^2
$V_{\alpha+2}^{3\cdot 2^r}$
$V_1^{2^r}+\mu$
$V_{\alpha+3}^{4\cdot 2^r}$
$V_{1+\alpha}^{2\cdot 2^r}+\mu$

Fig. 1

procedure until very few S_{2^r}-cycles having their vertices not contained in the last obtained cycle D remain. Then from D we construct a Hamilton cycle for G by an appropriate way. We give now the detail of this construction.

By induction, we will construct a sequence $C_0, C_1, C_2, C_3, \dots$ of cycles of G with the following properties:

Property (i): For an even index i, C_i contains all vertices of each of S_{2^r}-cycles $A^0, A^2, A^4, \dots, A^{3i}, A^{3i+2}, B^0, B^2, B^4, \dots, B^{3i}$ and B^{3i+2} and only these vertices. (All superscripts of A^ℓ and B^ℓ are always reduced modulo b.) Moreover, the edge

$$v_1^{(3i+3)2^r} v_2^{(3i+4)2^r}$$

of A^{3i+2} is an edge of C_i.

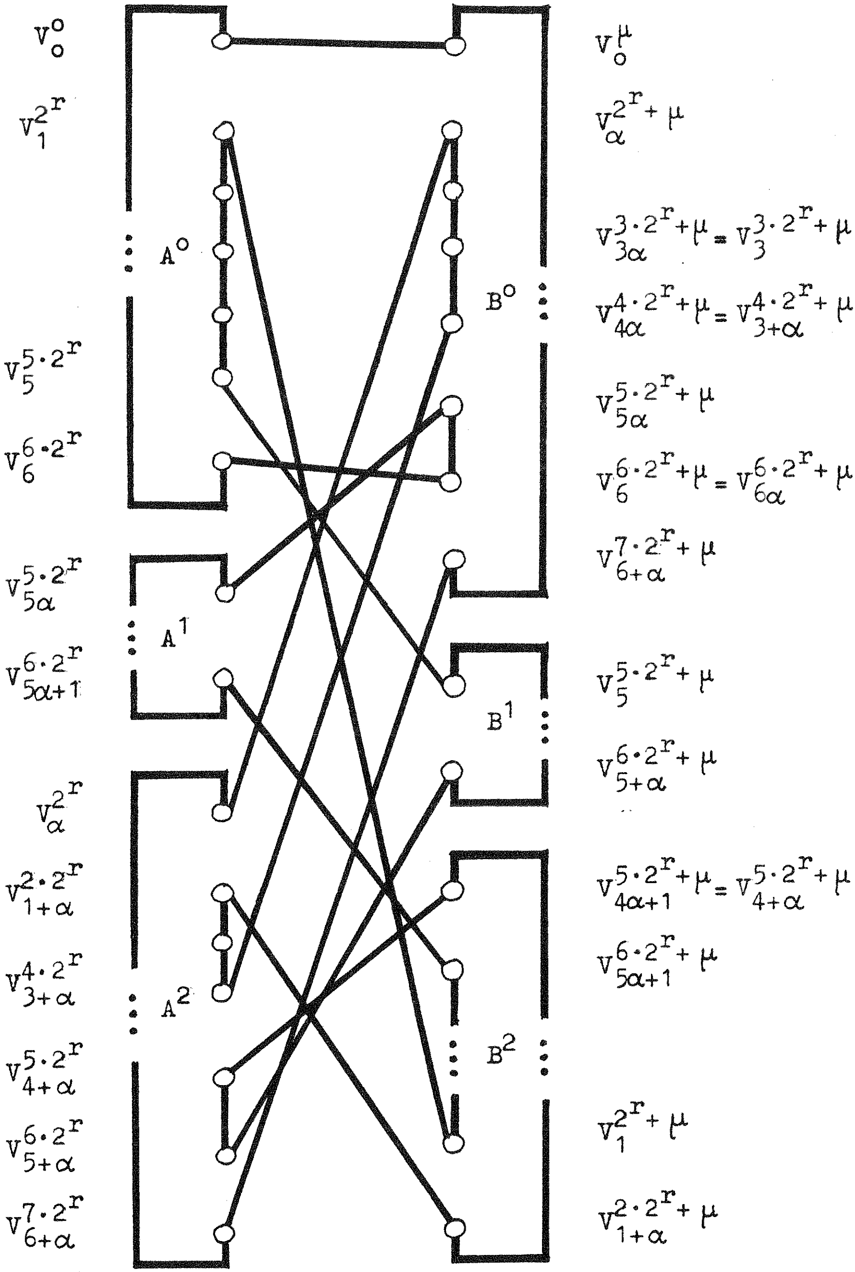
Property (ii): For an odd index i, C_i contains all vertices of each of S_{2^r}-cycles $A^0, A^2, A^4, \dots, A^{3(i+1)-2}, B^0, B^2, B^4, \dots, B^{3(i+1)-2}$ and $B^{3(i+1)}$ and only these vertices. (All superscripts of A^ℓ and B^ℓ are always reduced modulo b.) Moreover, the edge

$$v_0^{(3i+3)2^r+\mu} v_\alpha^{(3i+4)2^r+\mu}$$

of $B^{3(i+1)}$ is an edge of C_i.

The sequence of cycles $C_0, C_1, C_2, C_3, \dots$ is constructed as follows. First we take the alternating cycle $AC(v_0^\mu) = e_1 f_1 e_2 f_2 e_3 f_3 e_4 f_4$. Using Claim 1 and (3.9) it is not difficult to verify that e_1, e_2, e_3 and e_4 are edges of S_{2^r}-cycles B^0, A^2, B^2 and A^0, respectively. So from $AC(v_0^\mu)$ by replacing each $e_i, i = 1, 2, 3, 4$, by its S_{2^r}-complementing path $CP(e_i)$ we can obtain a cycle of G containing all vertices of each of A^0, A^2, B^0 and B^2 and only them. Since $b \geq 5$ and $b = \gcd(\mu, n)$ by (3.8), $\mu \geq 5$. So the edge $v_1^{3\cdot 2^r} v_2^{4\cdot 2^r}$ of A^2 is different from $e_2 = v_\alpha^{2^r} v_{\alpha+1}^{2\cdot 2^r}$. It follows that this edge is an edge of the obtained cycle. Thus, if we take this cycle as the cycle C_0 of the sequence, then it is clear that C_0 satisfies Property (i).

Let for an even index i the cycle C_i satisfying Property (i) have been constructed. Take the alternating cycle $AC(v_1^{(3i+3)2^r}) = e_1 f_1 e_2 f_2 e_3 f_3 e_4 f_4$. By the definition of $AC(v_j^i)$, (3.9) and Claim 1 it is not difficult to verify that e_1, e_2, e_3 and e_4 are edges of $A^{3i+2}, B^{3i+6}, A^{3i+4}$ and B^{3i+4}, respectively. By Property (i), e_1 is an edge of C_i. So if all vertices of each of B^{3i+6}, A^{3i+4} and B^{3i+4} are not contained in C_i, then from C_i by replacing the edge e_1 by the path $\overline{AP}(v_1^{(3i+3)2^r})$ we can get a cycle containing all vertices of each of $A^0, A^2, A^4, \dots, A^{3i+4}, B^0, B^2, B^4, \dots, B^{3i+4}$ and B^{3i+6} and only these vertices. Since $b \geq 5$ and $\gcd(\mu, n) = b$ by (3.8), we have $\mu \geq 5$. Hence it is not difficult to see that the edge $v_0^{(3i+6)2^r+\mu} v_\alpha^{(3i+7)2^r+\mu}$ of B^{3i+6} is different from $e_2 = v_2^{(3i+4)2^r+\mu} v_{2+\alpha}^{(3i+5)2^r+\mu}$. So this edge is an edge of the obtained



v_o^o
$v_1^{2^r}$
A^o
$v_5^{5\cdot 2^r}$
$v_6^{6\cdot 2^r}$
$v_{5\alpha}^{5\cdot 2^r}$
A^1
$v_{5\alpha+1}^{6\cdot 2^r}$
$v_\alpha^{2^r}$
$v_{1+\alpha}^{2\cdot 2^r}$
$v_{3+\alpha}^{4\cdot 2^r}$
A^2
$v_{4+\alpha}^{5\cdot 2^r}$
$v_{5+\alpha}^{6\cdot 2^r}$
$v_{6+\alpha}^{7\cdot 2^r}$
v_o^μ
$v_\alpha^{2^r+\mu}$
$v_{3\alpha}^{3\cdot 2^r+\mu} = v_3^{3\cdot 2^r+\mu}$
B^o
$v_{4\alpha}^{4\cdot 2^r+\mu} = v_{3+\alpha}^{4\cdot 2^r+\mu}$
$v_{5\alpha}^{5\cdot 2^r+\mu}$
$v_6^{6\cdot 2^r+\mu} = v_{6\alpha}^{6\cdot 2^r+\mu}$
$v_{6+\alpha}^{7\cdot 2^r+\mu}$
$v_5^{5\cdot 2^r+\mu}$
B^1
$v_{5+\alpha}^{6\cdot 2^r+\mu}$
$v_{4\alpha+1}^{5\cdot 2^r+\mu} = v_{4+\alpha}^{5\cdot 2^r+\mu}$
$v_{5\alpha+1}^{6\cdot 2^r+\mu}$
B^2
$v_1^{2^r+\mu}$
$v_{1+\alpha}^{2\cdot 2^r+\mu}$

Fig. 2

cycle. We take this cycle as the cycle C_{i+1} of the sequence. Then it is clear that C_{i+1} satisfies Property (ii).

Now let for an odd index i the cycle C_i satisfying Property (ii) have been constructed. Take the alternating cycle $AC(v_0^{(3i+3)2^r+\mu}) = e_1 f_1 e_2 f_2 e_3 f_3 e_4 f_4$. Then as before it is not difficult to verify that e_1, e_2, e_3 and e_4 are edges of $B^{3(i+1)}$, $A^{3(i+1)+2}$, $B^{3(i+1)+2}$ and $A^{3(i+1)}$, respectively. By Property (ii), e_1 is an edge of C_i. So if all vertices of each of $A^{3(i+1)+2}$, $B^{3(i+1)+2}$ and $A^{3(i+1)}$ are not contained in C_i, then from C_i by replacing the edge e_1 by the path $\overline{AP}(v_0^{(3i+3)2^r+\mu})$ we can get a cycle containing all vertices of each of $A^0, A^2, A^4, \dots, A^{3(i+1)}, A^{3(i+1)+2}, B^0, B^2, B^4, \dots, B^{3(i+1)}$ and $B^{3(i+1)+2}$ and only these vertices. Since $b \geq 5$, as before, it is not difficult to see that the edge $v_1^{(3i+6)2^r} v_2^{(3i+7)2^r}$ of $A^{3(i+1)+2}$ is different from e_2. So this edge is an edge of the obtained cycle. Take this cycle as the cycle C_{i+1} of the sequence. Then C_{i+1} satisfies Property (i).

Note that the number of S_{2^r}-cycles all vertices of which are contained in a cycle C_i of the constructed sequence is 4 + 3i. Therefore, we have the following three possibilities to consider.

(2.3.1) 2b = (4 + 3t) + 2 for some positive integer t.

Since $b \geq 5$ is odd and $t = (2b-6)/3$, $t \geq 4$ is even and b must be divisible by 3. It is not difficult to see that we can construct the cycle C_{t-1}. Since $t-1 = (2b-9)/3$ is odd, by Property (ii) all vertices of each of $A^0, A^2, A^4, \dots, A^{b-1}, A^1, A^3, \dots, A^{b-10}, A^{b-8}, B^0, B^2, B^4, \dots, B^{b-1}, B^1, B^3, \dots, B^{b-10}, B^{b-8}$ and B^{b-6} are contained in C_{t-1}. The remaining vertices of G not contained in C_{t-1} are vertices of $A^{b-6}, A^{b-4}, A^{b-2}, B^{b-4}$ and B^{b-2}.

To facilitate understanding what follows the reader is advised to make himself a drawing of a cycle C_i and a path $\overline{AP}(v_y^x)$ (with all three S_{2^r}-complementing paths contained in it) when a cycle C_{i+1} is obtained from C_i by replacing the edge $v_y^x v_{y+\alpha^x}^{x+2^r}$ of C_i by the path $\overline{AP}(v_y^x)$.

Take the vertex $v_{\alpha-1}^{(b+\alpha-3)2^r}$ of A^{b-2} and consider the alternating cycle $AC(v_{\alpha-1}^{(b+\alpha-3)2^r}) = e_1 f_1 e_2 f_2 e_3 f_3 e_4 f_4$ (Fig. 3). By Claim 1 and the definition of an alternating cycle $AC(v_j^i)$, it is not difficult to verify that e_1, e_2, e_3 and e_4 are edges of A^{b-2}, B^{b-4}, A^0 and B^{b-6}, respectively, and both e_3 and e_4 are edges of C_{t-1}. We determine in what order the vertices $v_{2\alpha-1}^{(b+\alpha-2)2^r}$ and $v_{2\alpha}^{(b+\alpha-1)2^r}$ incident with e_3 and the vertices $v_{(\alpha-1)}^{((b+\alpha-3)2^r+\mu)}$ and $v_{(2\alpha-1)}^{((b+\alpha-2)2^r+\mu)}$ incident with e_4 lie in C_{t-1}. For this we follow each cycle $C_i, i = 0, 1, 2, \dots,$ by starting at v_0^0 and then going in the direction from v_0^0 to v_0^μ. It is clear from the constructions of C_i that if a vertex v_y^x appears before a vertex v_w^z in C_i and $i < j$, then v_y^x also appears before v_w^z in C_j.

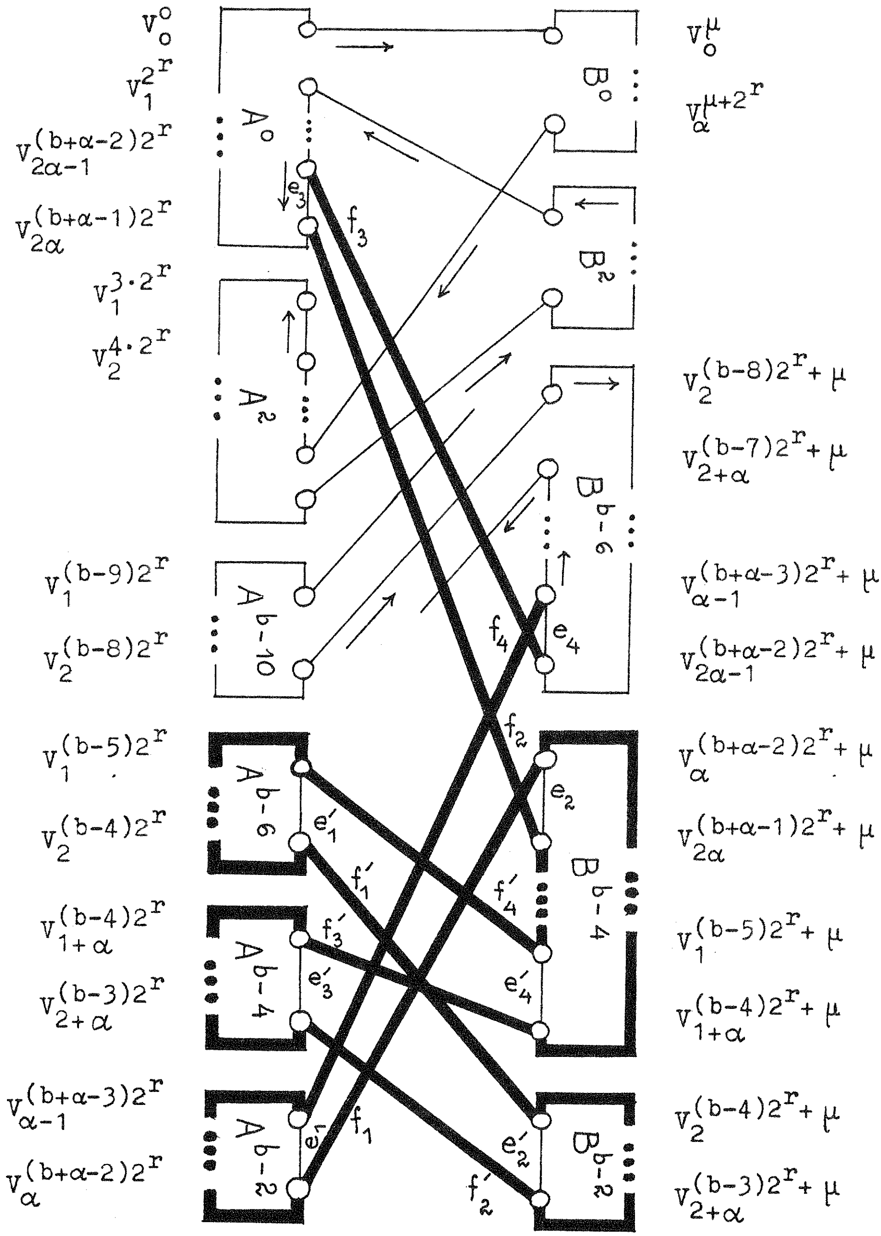

v_0^0
$v_1^{2^r}$
$v_{2\alpha-1}^{(b+\alpha-2)2^r}$
$v_{2\alpha}^{(b+\alpha-1)2^r}$
$v_1^{3\cdot 2^r}$
$v_2^{4\cdot 2^r}$
$v_1^{(b-9)2^r}$
$v_2^{(b-8)2^r}$
$v_1^{(b-5)2^r}$
$v_2^{(b-4)2^r}$
$v_{1+\alpha}^{(b-4)2^r}$
$v_{2+\alpha}^{(b-3)2^r}$
$v_{\alpha-1}^{(b+\alpha-3)2^r}$
$v_\alpha^{(b+\alpha-2)2^r}$
A^0
A^2
A^{b-10}
A^{b-6}
A^{b-4}
A^{b-2}
B^0
B^2
B^{b-6}
B^{b-4}
B^{b-2}
e_1 e_2 e_3 e_4
f_1 f_2 f_3 f_4
e_1' e_2' e_3' e_4'
f_1' f_2' f_3' f_4'
v_0^μ
$v_\alpha^{\mu+2^r}$
$v_2^{(b-8)2^r+\mu}$
$v_{2+\alpha}^{(b-7)2^r+\mu}$
$v_{\alpha-1}^{(b+\alpha-3)2^r+\mu}$
$v_{2\alpha-1}^{(b+\alpha-2)2^r+\mu}$
$v_\alpha^{(b+\alpha-2)2^r+\mu}$
$v_{2\alpha}^{(b+\alpha-1)2^r+\mu}$
$v_1^{(b-5)2^r+\mu}$
$v_{1+\alpha}^{(b-4)2^r+\mu}$
$v_2^{(b-4)2^r+\mu}$
$v_{2+\alpha}^{(b-3)2^r+\mu}$

Fig. 3

Since $v_{2\alpha}^{(b+\alpha-1)2^r} \neq v_0^0$, it is not difficult to verify that $v_{2\alpha-1}^{(b+\alpha-2)2^r}$ appears before $v_{2\alpha}^{(b+\alpha-1)2^r}$ in C_0 (Fig. 3). By the remark at the end of the preceding paragraph, $v_{2\alpha-1}^{(b+\alpha-2)2^r}$ also appears before $v_{2\alpha}^{(b+\alpha-1)2^r}$ in C_{t-1}.

For any even index $i < t$ consider the edge $v_1^{(3i+3)2^r} v_2^{(3i+4)2^r}$ of A^{3i+2}. By Property (i) this edge is an edge of C_i. We prove now by induction on i that the vertex $v_2^{(3i+4)2^r}$ incident with this edge appears before $v_1^{(3i+3)2^r}$ in C_i. In C_0, it is easy to verify that $v_2^{4\cdot 2^r}$ appears before $v_1^{3\cdot 2^r}$. (These vertices are vertices of A^2.) Suppose that for an even index $i < t$ such that $i+2 < t$, the vertex $v_2^{(3i+4)2^r}$ has been proved to appear before $v_1^{(3i+3)2^r}$ in C_i. Since the cycle C_{i+1} is obtained from C_i by replacing the edge $v_1^{(3i+3)2^r} v_2^{(3i+4)2^r}$ of C_i by the path $\overline{AP}(v_1^{(3i+3)2^r})$ containing the vertices $v_0^{((3i+6)2^r+\mu)}$ and $v_\alpha^{((3i+7)2^r+\mu)}$ of B^{3i+6}, we can easily see that $v_\alpha^{((3i+7)2^r+\mu)}$ appears before $v_0^{((3i+6)2^r+\mu)}$ in C_{i+1}. In its turn, C_{i+2} is obtained from C_{i+1} by replacing the edge $v_0^{((3i+6)2^r+\mu)} v_\alpha^{((3i+7)2^r+\mu)}$ by the path $\overline{AP}(v_0^{((3i+6)2^r+\mu)})$ containing the vertices $v_1^{(3i+9)2^r}$ and $v_2^{(3i+10)2^r}$ of A^{3i+8}. Therefore, it is also easily seen that $v_2^{(3i+10)2^r}$ appears before $v_1^{(3i+9)2^r}$ in C_{i+2}. The assertion has been proved.

Since $2b = (4+3t)+2$, we have $t-2 = (2b-12)/3$ is even. So $3(t-2)+2 \equiv b-10 \pmod b$ and the cycle C_{t-2} contains all vertices of each of $A^0, A^2, A^4, \dots, A^{b-1}, A^1, A^3, \dots, A^{b-10}, B^0, B^2, B^4, \dots, B^{b-1}, B^1, B^3, \dots, B^{b-10}$. By the assertion proved in the preceding paragraph, the vertex $v_2^{(3(t-2)+4)2^r} = v_2^{(b-8)2^r}$ appears before $v_1^{(3(t-2)+3)2^r} = v_1^{(b-9)2^r}$ in C_{t-2}. Since C_{t-1} is obtained from C_{t-2} by replacing the edge $v_1^{(b-9)2^r} v_2^{(b-8)2^r}$ by the path $\overline{AP}(v_1^{(b-9)2^r})$ containing the vertices $v_{2\alpha-1}^{((b+\alpha-2)2^r+\mu)}$ and $v_{\alpha-1}^{((b+\alpha-3)2^r+\mu)}$ of B^{b-6}, it is easily checked (Fig. 3) that the vertex $v_{2\alpha-1}^{((b+\alpha-2)2^r+\mu)}$ appears before $v_{\alpha-1}^{((b+\alpha-3)2^r+\mu)}$ in C_{t-1}. Thus, the order in which the vertices $v_{2\alpha-1}^{(b+\alpha-2)2^r}, v_{2\alpha}^{(b+\alpha-1)2^r}, v_{\alpha-1}^{((b+\alpha-3)2^r+\mu)}$ and $v_{2\alpha-1}^{((b+\alpha-2)2^r+\mu)}$ lie in C_{t-1} are as shown in Figure 4.

By the definition of the alternating cycle $AC(v_{\alpha-1}^{(b+\alpha-3)2^r}) = e_1 f_1 e_2 f_2 e_3 f_3 e_4 f_4$, the edge f_3 connects the vertex $v_{2\alpha-1}^{(b+\alpha-2)2^r}$ with the vertex $v_{(2\alpha-1)}^{((b+\alpha-2)2^r+\mu)}$. On the other hand, for the vertex $v_1^{(b-5)2^r}$ of A^{b-6}, let $AC(v_1^{(b-5)2^r}) = e_1' f_1' e_2' f_2' e_3' f_3' e_4' f_4'$ (Fig. 3). Then e_1', e_2', e_3' and e_4' are edges of $A^{b-6}, B^{b-2}, A^{b-4}$ and B^{b-4}, respectively. Form the path

$$Q = f_2 Q_1 f_4' CP(e_1') f_1' CP(e_2') f_2' CP(e_3') f_3' Q_2 f_1 CP(e_1) f_4,$$

where Q_1 and Q_2 are the subpaths of B^{b-4} not containing both e_2 and e_4' and connecting the vertices incident with f_2 and f_4' and with f_3' and f_1, respectively (Fig. 3). Then Q connects the vertex $v_{2\alpha}^{(b+\alpha-1)2^r}$ with the vertex $v_{(\alpha-1)}^{((b+\alpha-3)2^r+\mu)}$. It is not difficult to verify that every vertex of Q except its endvertices is a vertex of one of $A^{b-2}, A^{b-4}, A^{b-6}, B^{b-2}$ or B^{b-4}, and conversely, every vertex of each of

$A^{b-2}, A^{b-4}, A^{b-6}, B^{b-2}$ and B^{b-4} is contained in Q. Therefore, G has the following Hamilton cycle C (Fig. 4). Start C at the vertex $v_{2\alpha}^{(b+\alpha-1)2^r}$ and go around C_{t-1} in the chosen direction until reaching $v_{(2\alpha-1)}^{((b+\alpha-2)2^r+\mu)}$. Now take the edge f_3 to $v_{2\alpha-1}^{(b+\alpha-2)2^r}$ and again go around C_{t-1} but in the direction opposite to the chosen direction until reaching $v_{(\alpha-1)}^{((b+\alpha-3)2^r+\mu)}$. Finally go along the path Q to return to $v_{2\alpha}^{(b+\alpha-1)2^r}$.

(2.3.2) 2b = (4 + 3t) + 1 for some positive integer t.

Since $b \geq 5$ is odd and $t = (2b-5)/3$, $t \geq 3$ and it is odd. Also, the cycle C_{t-1} can be constructed. Since $t-1 = 2(b-4)/3$ is even, by Property (i), the cycle C_{t-1} contains all vertices of each of S_{2^r}-cycles $A^0, A^2, A^4, \ldots, A^{b-1}, A^1, A^3, \ldots, A^{b-8}$, $A^{b-6}, B^0, B^2, B^4, \ldots, B^{b-1}, B^1, B^3, \ldots, B^{b-8}$ and B^{b-6}. The remaining vertices of G not contained in C_{t-1} are vertices of $A^{b-4}, A^{b-2}, B^{b-4}$ and B^{b-2}.

Take the vertices $v_0^{(b-4)2^r}$ and $v_2^{(b-2)2^r}$ of A^{b-4} and consider the alternating cycles $AC(v_0^{(b-4)2^r}) = e_1 f_1 e_2 f_2 e_3 f_3 e_4 f_4$ and $AC(v_2^{(b-2)2^r}) = e'_1 f'_1 e'_2 f'_2 e'_3 f'_3 e'_4 f'_4$ (Fig. 5). By definition, we see that e_1, e_2, e_3 and e_4 are edges of $A^{b-4}, B^{b-2}, A^{b-2}$ and B^{b-4}, respectively. Similarly, e'_1, e'_2, e'_3 and e'_4 are edges of A^{b-4}, B^2, A^{b-2} and B^0, respectively. Now we form paths P_1 and P_2 of G as follows. Start P_1 with the subpath $f'_4(v_2^{(b-2)2^r} v_1^{(b-3)2^r})f_1$. Then take the S_{2^r}-complementing path $CP(e_2)$. The last subpath of P_1 is $f_2(v_{\alpha+1}^{(b-2)2^r} v_{\alpha+2}^{(b-1)2^r})f'_3$. Start P_2 with the subpath $f'_1(v_3^{(b-1)2^r} v_4^{b2^r} \ldots v_{n-1}^{(b-5)2^r} v_0^{(b-4)2^r})f_4$. Then take the S_{2^r}-complementing path $CP(e_4)$. The last subpath of P_2 is $f_3(v_\alpha^{(b-3)2^r} v_{\alpha-1}^{(b-4)2^r} v_{\alpha-2}^{(b-5)2^r} \ldots v_{\alpha+4}^{(b+1)2^r} v_{\alpha+3}^{b2^r})f'_2$.

By the constructions of P_1 and P_2, it is clear that P_1 and P_2 are disjoint, all vertices of each of $A^{b-4}, A^{b-2}, B^{b-4}$ and B^{b-2} are contained in either P_1 or P_2 and only vertices of P_1 and P_2 contained in C_{t-1} are their endvertices. Further, the endvertices of P_1 are the vertices incident with e'_4 and the endvertices of P_2 are the vertices incident with e'_2. It is also not difficult to show that e'_4 and e'_2 are edges of C_{t-1}. Therefore, from C_{t-1} by replacing e'_4 by P_1 and e'_2 by P_2 we get a Hamilton cycle of G.

(2.3.3) 2b = 4 + 3t for some positive integer t.

Recall that $b \geq 5$ is odd. Since $t = (2b-4)/3$, $t \geq 2$ and it is even. By Properties (i) and (ii) of C_i, it is not difficult to see that we can construct the cycle C_t which contains all vertices of all S_{2^r}-cycles of G. This means that C_t is a Hamilton cycle of G.

The proof of Theorem 1 is complete. □

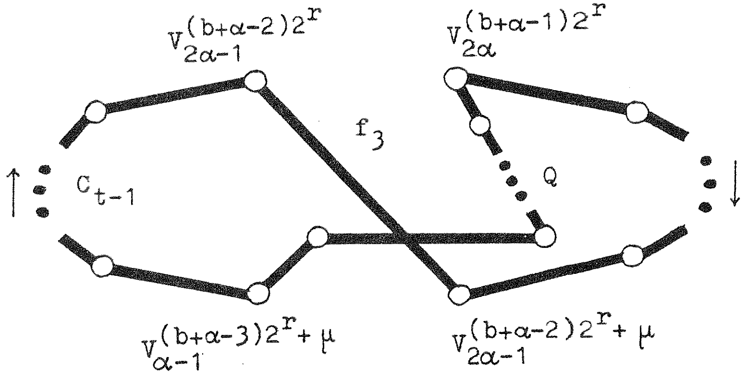

$v_{2\alpha-1}^{(b+\alpha-2)2^r}$
$v_{2\alpha}^{(b+\alpha-1)2^r}$
f_3
c_{t-1}
Q
$v_{\alpha-1}^{(b+\alpha-3)2^r+\mu}$
$v_{2\alpha-1}^{(b+\alpha-2)2^r+\mu}$

Fig. 4

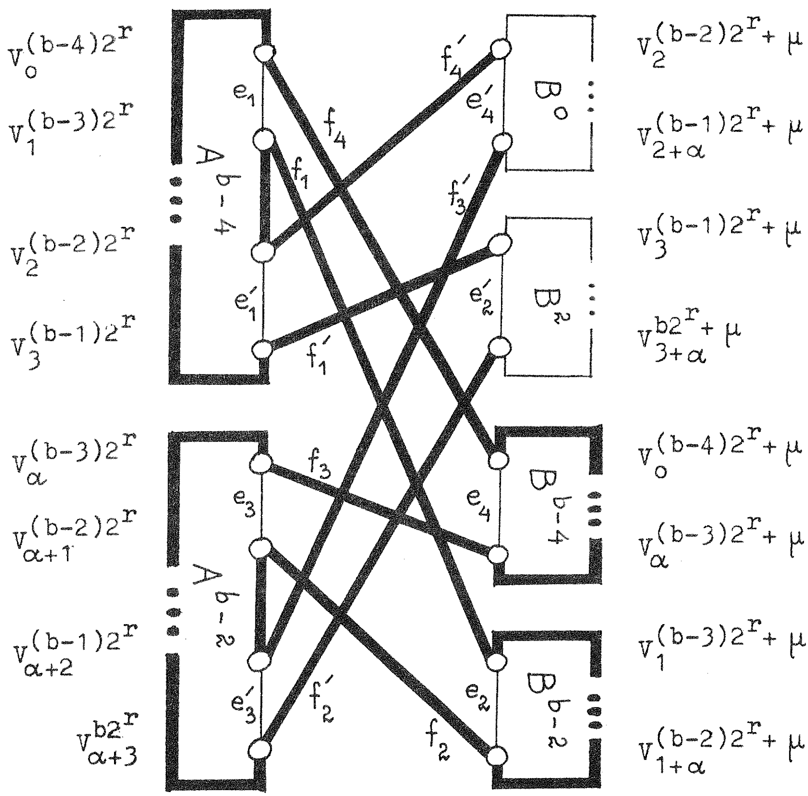

$v_0^{(b-4)2^r}$
$v_1^{(b-3)2^r}$
$v_2^{(b-2)2^r}$
$v_3^{(b-1)2^r}$
$v_\alpha^{(b-3)2^r}$
$v_{\alpha+1}^{(b-2)2^r}$
$v_{\alpha+2}^{(b-1)2^r}$
$v_{\alpha+3}^{b2^r}$
A^{b-4}
A^{b-2}
e_1
e_1'
e_3
e_3'
f_1
f_1'
f_2
f_2'
f_3
f_3'
f_4
f_4'
e_4'
e_2'
e_4
e_2
B^0
B^2
B^{b-4}
B^{b-2}
$v_2^{(b-2)2^r+\mu}$
$v_{2+\alpha}^{(b-1)2^r+\mu}$
$v_3^{(b-1)2^r+\mu}$
$v_{3+\alpha}^{b2^r+\mu}$
$v_0^{(b-4)2^r+\mu}$
$v_\alpha^{(b-3)2^r+\mu}$
$v_1^{(b-3)2^r+\mu}$
$v_{1+\alpha}^{(b-2)2^r+\mu}$

Fig. 5

As an application of Theorem 1, we prove now the following result which is a partial affirmative answer to the question whether all connected cubic (m,n)-metacirculant graphs, other than the Petersen graph, have a Hamilton cycle.

Theorem 2. *Let m and n be positive integers such that every odd prime divisor of m is not a divisor of $\varphi(n)$ where φ is the Euler φ-function. Then every connected cubic (m,n)-metacirculant graph possesses a Hamilton cycle.*

Proof. Let m and n satisfy the hypotheses of Theorem 2 and let $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ be a connected cubic (m,n)-metacirculant graph. If m is odd or m = 2 or m is divisible by 4, then by the results obtained in [4, 6, 10] G possesses a Hamilton cycle. Therefore, we may assume from now on that $m > 2$ is even and not divisible by 4. Suppose that G is isomorphic to the Petersen graph. Then mn = 10 because the orders of G and the Petersen graph are equal to mn and 10, respectively. Since $m > 2$ is even, this implies that m = 10, n = 1. It is clear that for these values of m and n G is a Cayley graph, contradicting the fact that the Petersen graph is not a Cayley graph. Thus, G is not isomorphic to the Petersen graph. So if $S_0 \neq \emptyset$, then G again has a Hamilton cycle by [6]. Therefore, we also may assume from now on that $S_0 = \emptyset$. Since G is a cubic (m,n)-metacirculant graph, this implies that only the following may happen:

(i) $S_0 = \emptyset, S_i = \{s\}$ with $0 \leq s < n$ for some $i \in \{1, 2, \ldots, \mu - 1\}, S_j = \emptyset$ for all $i \neq j \in \{1, 2, \ldots, \mu - 1\}$ and $S_\mu = \{k\}$ with $0 \leq k < n$;

(ii) $S_0 = \cdots = S_{\mu-1} = \emptyset$ and $|S_\mu| = 3$.

Since G is connected and $m > 2$ is even, (ii) cannot occur. So only (i) may happen. By Lemma 2, without loss of generality, we may assume that such a graph G has one of the following forms:

1. $S_0 = \emptyset, S_1 = \{s\}, S_2 = \cdots = S_{\mu-1} = \emptyset$ and $S_\mu = \{k\}$;

2. $S_0 = S_1 = \cdots = S_{2^r-1} = \emptyset$ for some $r \geq 1, S_{2^r} = \{s\}, S_{2^r+1} = \cdots = S_{\mu-1} = \emptyset$ and $S_\mu = \{k\}$.

In both cases 1 and 2, by Lemma 3,

$$gcd(k, s(1 + \alpha + \alpha^2 + \cdots + \alpha^{\mu-1}), n) = 1. \qquad (3.12)$$

On the other hand, by the definition of (m,n)-metacirculant graphs, we have

I. $\alpha^{2\mu}s \equiv s \pmod n$

$$\iff (\alpha^\mu + 1)(\alpha - 1)(1 + \alpha + \alpha^2 + \cdots + \alpha^{\mu-1})s \equiv 0 \ (\bmod\ n), \text{ and} \qquad (3.13)$$

$$\iff (\alpha^\mu + 1)k \equiv 0 \ (\bmod n). \qquad (3.14)$$

Let $z = n/gcd(\alpha^\mu + 1, n)$. Then z is a divisor of both k and $(\alpha - 1)(1 + \alpha + \cdots + \alpha^{\mu-1})s$. Therefore, by (3.12) z is a divisor of $\alpha - 1$. Thus,

$$(\alpha^\mu + 1)(\alpha - 1) = (\alpha + 1)(1 - \alpha + \alpha^2 - \cdots + \alpha^{\mu-1})(\alpha - 1) \equiv 0 \ (\bmod n). \quad (3.15)$$

It follows that $(\alpha^m - 1) = (\alpha^\mu + 1)(\alpha - 1)(1 + \alpha + \alpha^2 + \cdots + \alpha^{\mu-1}) \equiv 0 \ (\bmod n)$, i.e., the order of α in Z_n^* is a divisor of m. But it is well-known that $|Z_n^*| = \varphi(n)$. So by the hypotheses of our theorem, it follows that $\alpha^2 \equiv 1 \ (\bmod n)$. By Theorem 1, G possesses a Hamilton cycle. This completes the proof of Theorem 2. □

The hypotheses of Theorem 2 become simple when m has only one odd prime divisor. For such values of m, it seems that the problem of the existence of a Hamilton cycle in connected cubic (m,n)-metacirculant graphs would be easier to solve than for other values of m. Because of this we reformulate Theorem 2 for these values of m in the following corollary.

Corollary 3. *Let $m = 2^a p^b$ with p an odd prime and n be such that $\varphi(n)$ is not divisible by p. Then every connected cubic (m,n)-metacirculant graph has a Hamilton cycle.*

The following result also might be useful in considering the problem of the existence of a Hamilton cycle in connected cubic (m.n)-metacirculant graphs. Since connected cubic (m,n)-metacirculant graphs have been proved to be hamiltonian for m odd [6], m = 2 [4, 6] and m divisible by 4 [10], we may assume in the following theorem that m is even, greater than 2 and not divisible by 4.

Theorem 4. *Let m be even, greater than 2 and not divisible by 4 and $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ be a connected cubic (m,n)-metacirculant graph. Then G possesses a Hamilton cycle if one of the numbers $(\alpha + 1)$ or $(1 - \alpha + \alpha^2 - \cdots - \alpha^{\mu-2} + \alpha^{\mu-1})$ is relatively prime to n.*

Proof. Let the hypotheses of Theorem 4 be satisfied. Suppose that G is isomorphic to the Petersen graph. Then mn = 10 because the orders of G and the Petersen graph are equal to mn and 10, respectively. Since m is even and greater than 2, this implies that m = 10, n = 1. It is clear that for these values of m and n the graph G is a Cayley graph, contradicting the fact that the Petersen graph is not a Cayley graph. Thus, G is not isomorphic to the Petersen graph. So if $S_0 \neq \emptyset$, then G has a Hamilton cycle by [6]. Therefore, we assume from now on that $S_0 = \emptyset$.

Since G is a cubic (m,n)-metacirculant graph, this implies that only the following may happen:

(i) $S_0 = \emptyset, S_i = \{s\}$ with $0 \leq s < n$ for some $i \in \{1, 2, \ldots, \mu - 1\}, S_j = \emptyset$ for all $i \neq j \in \{1, 2, \ldots, \mu - 1\}$ and $S_\mu = \{k\}$ with $0 \leq k < n$;

(ii) $S_0 = \cdots = S_{\mu-1} = \emptyset$ and $|S_\mu| = 3$.

Since G is connected and $m > 2$ is even, (ii) cannot occur. So only (i) may happen. By Lemma 2, without loss of generality we may assume that G has one of the following forms:

1. $S_0 = \emptyset, S_1 = \{s\}, S_2 = \cdots = S_{\mu-1} = \emptyset$ and $S_\mu = \{k\}$;

2. $S_0 = \cdots = S_{2^r-1} = \emptyset$ for some $r \geq 1, S_{2^r} = \{s\}, S_{2^r+1} = \cdots = S_{\mu-1} = \emptyset$ and $S_\mu = \{k\}$.

We consider these possibilities in turn.

Case 1. $S_0 = \emptyset, S_1 = \{s\}, S_2 = \cdots = S_{\mu-1} = \emptyset$ and $S_\mu = \{k\}$.

Let ρ be the automorphism of G defined by $\rho(v_j^i) = v_{j+1}^i$. Then ρ is semiregular. Therefore, $\rho^{\alpha-1}$ is also semiregular and we can construct the quotient graph $G/\rho^{\alpha-1}$. It is not difficult to verify that $G/\rho^{\alpha-1}$ is isomorphic to the cubic (m,a)-metacirculant graph $G' = MC(m, a, \alpha', S'_0, S'_1, \ldots, S'_\mu)$, where $a = \gcd(\alpha-1, n), 1 = \alpha' \equiv \alpha \pmod{a}, S'_0 = \emptyset, S'_1 = \{s'\}$ with $s' \equiv s \pmod{a}, S'_2 = \cdots = S'_{\mu-1} = \emptyset$ and $S'_\mu = \{k'\}$ with $k' \equiv k \pmod{a}$. Therefore, we can identify these two graphs.

First assume that $\alpha + 1$ is relatively prime to n. If n is even, then G has a Hamilton cycle [9, Lemma 6]. If n is odd, then we can construct a Hamilton cycle C of G' as in the proof of the main theorem in [10]. The path P of coil(C), which starts at v_0^0, terminates at v_f^0 with $f \equiv (\alpha - 1)d \pmod{n}$, where

$$d = -[k - s(1 + \alpha + \cdots + \alpha^{\mu-1})](1 + \alpha + \cdots + \alpha^\mu).$$

(The reader is referred to [10] for all these details.) Let $c = \gcd(\alpha^\mu + 1, n)$. By [10, Lemma 4], n = ac. Therefore, the order t of $\rho^{\alpha-1}$ is $n/a = c = \gcd(\alpha^\mu + 1, n) = \gcd((\alpha + 1)(1 - \alpha + \alpha^2 - \cdots + \alpha^{\mu-1}), n)$. Since $\gcd(\alpha + 1, n) = 1$, it follows that $c = \gcd(1 - \alpha + \alpha^2 - \cdots + \alpha^{\mu-1}, n)$.

We have $(1 + \alpha + \alpha^2 + \cdots + \alpha^\mu) = (1 + \alpha)(1 + \alpha^2 + \alpha^4 + \cdots + \alpha^{\mu-1})$. If p is an (odd) divisor of $g = \gcd(1 + \alpha + \cdots + \alpha^\mu, c)$, then p is a divisor of both $(1 + \alpha^2 + \alpha^4 + \cdots + \alpha^{\mu-1})$ and $(1 - \alpha + \alpha^2 - \cdots + \alpha^{\mu-1})$ because $\gcd(\alpha + 1, n) = 1$. Therefore, p is a divisor of $\alpha + \alpha^3 + \alpha^5 \cdots + \alpha^{\mu-2} = \alpha(1 + \alpha^2 + \alpha^4 + \cdots + \alpha^{\mu-3})$. Since $\gcd(\alpha, n) = 1$, it follows that p is a divisor of $(1 + \alpha^2 + \alpha^4 + \cdots + \alpha^{\mu-3})$. So p is a divisor of $\alpha^{\mu-1}$, contradicting $\gcd(\alpha^{\mu-1}, n) = 1$. Thus, $\gcd(1 + \alpha + \cdots + \alpha^\mu, c) = 1$.

On the other hand, by Lemma 3, $\gcd([k - s(1+\alpha+\cdots+\alpha^{\mu-1})], n) = 1$. So gcd(d,c) = gcd(d,t) = 1. By Lemma 1, G has a Hamilton cycle in this subcase.

Now assume that $\gcd(1-\alpha+\alpha^2-\cdots+\alpha^{\mu-1}, n) = 1$. This implies by (3.15) that $\alpha^2 \equiv 1 \pmod{n}$. By Theorem 1, G again possesses a Hamilton cycle in this subcase.

Case 2. $S_0 = \cdots = S_{2^r-1} = \emptyset$ for some $r \geq 1, S_{2^r} = \{s\}, S_{2^r+1} = \cdots = S_{\mu-1} = \emptyset$ and $S_\mu = \{k\}$.

Let $a = \gcd(\alpha - 1, n), b = \gcd(1-\alpha+\alpha^2-\cdots+\alpha^{\mu-1}, n)$. By (3.15), n/(ab) is a divisor of $\gcd(\alpha+1, n)$. Therefore, if $\gcd(\alpha+1,n) = 1$, then n/(ab) = 1 and $\gcd(n/(ab), \mu a - 1) = 1$. By Lemma 4(i), G has a Hamilton cycle in this subcase. If $b = \gcd(1-\alpha+\alpha^2-\cdots+\alpha^{\mu-1}, n) = 1$, then by Lemma 4(ii), G again has a Hamilton cycle.

The proof of Theorem 4 is complete. □

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(Received 23/1/96)