A Local Neighbourhood Condition for n-extendable Graphs

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ABSTRACT

Let G be a connected graph with even order. Let v $\in$ V(G). We define $N_k(v) = \{u | u \in V(G)$ and d(u,v) = k$\}$. It is proved that if for each vertex v $\in$ V(G) and for each independent set $S \subseteq N_2(v)$, $|N(v) \cap N(S)| \geq |S| + 2n$, then G is n-extendable. Several previously known sufficient conditions for n-extendable graphs follow as corollaries.

All graphs in this paper are finite, undirected and simple.

Let G be a graph of order ν with a perfect matching and let n be a positive integer such that $n \leq (\nu - 2)/2$. G is said to be n-extendable if G has n independent edges and any n independent edges of G are contained in a perfect matching of G.

Let G be a connected graph and let u and v be a pair of vertices of G such that d(u,v) = 2. We use I(u,v) to denote $|N(u) \cap N(v)|$. We define the divergence $\alpha^*(u, v)$ as follows:

$n_{u,v}(w) = \max \{|S| \,|w \in N(u) \cap N(v)$, S is an independent set in $G[\{w\} \cup N_G(w)]$ containing u and v $\}$.

$\alpha^*(u, v) = max_w\{n_{u,v}(w) | w \in N(u) \cap N(v)\}$.

Let v be a vertex of G. We define $N_k(v) = \{u | u \in V(G)$ and d(u,v) = k $\}$. We denote by $\omega(G)$ the number of components of G and by o(G) the number of odd components of G.

All terminology and notation not defined in this paper are from [2].

Since Plummer [7] introduced the concept of n-extendable graphs in 1980, much work has been done on this topic (for example, see [1], [3], [4] and [8]). In [5], Lou introduced a sufficient condition for n-extendable graphs in terms of the divergence and some other sufficient conditions for n-extendable graphs. However there are not many known general sufficient conditions for n-extendable

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graphs at present. In this paper we introduce a new sufficient condition for n-extendable graphs, which implies the divergence condition and a degree condition set up by Plummer [7]. Lou [6] also gave this type of sufficient condition for hamiltonian graphs. The following is the main result of this paper.

THEOREM 1. *Let G be a connected graph and let $k \geq 0$ be an integer. If, for each vertex $v \in V(G)$ and for each independent set $R \subseteq N_2(v)$, $|N(v) \cap N(R)| \geq |R| + k + 1$, then for each subset $S \subseteq V(G)$, $\omega(G - S) \leq |S| - k$.*

PROOF: Let $S \subseteq V(G)$ and let $|S| = s$. Let $\omega(G-S) = t$ and let $C_1, C_2, \ldots, C_t$ be the components of $G - S$. Let $S = \{v_1, v_2, \ldots, v_s\}$ and let k_i be the number of components in $G - S$ which are adjacent to v_i $(i = 1, 2, \ldots, s)$. Without loss of generality, assume $k_1 \leq k_2 \leq \ldots \leq k_s$. Let $k_{m_j} = \max\{k_i \mid v_i$ is adjacent to C_j and $1 \leq i \leq s\}$ $(j = 1, 2, \ldots, t)$. Without loss of generality, assume $k_{m_1} \leq k_{m_2} \leq \cdots \leq k_{m_t}$. We choose $S \subseteq V(G)$ such that $|S| - \omega(G-S)$ is as small as possible.

Claim 1: We have $k_i \geq 2$ for $1 \leq i \leq s$.

Suppose this is not the case. Then there is k_i $(1 \leq i \leq s)$ such that $k_i \leq 1$. We replace S by $S' = S \backslash \{v_i\}$. Then $\omega(G - S') \geq \omega(G - S)$. However $|S'| = |S| - 1$. So $|S'| - \omega(G - S') < |S| - \omega(G - S)$, contradicting the choice of S.

Assume that v_i is adjacent to a vertex u in a component C of G-S. We define the set T to consist of the k_i vertices each of which is adjacent to v_i and is chosen respectively from one of the k_i components which are adjacent to v_i. Any two vertices in T belong to two different components of G-S. Then $T \cap V(C) = \{u\}$, $T \backslash \{u\} \subseteq N_2(u)$ and $T \backslash \{u\}$ is an independent set. So $N(u) \cap N(T \backslash \{u\}) \subseteq S$. By the hypotheses of this theorem, $|N(u) \cap N(T \backslash \{u\})| \geq |T \backslash \{u\}| + k + 1$. So u is adjacent to at least $|T \backslash \{u\}| + k + 1 = k_i + k$ vertices in S. For each component adjacent to v_i, the component is adjacent to at least $k_i + k$ vertices in S. Considering all vertices in S which are adjacent to component C_j, we know that C_j is adjacent to at least $k_{m_j} + k$ vertices in S. For the convenience of explanation, if a vertex in S is adjacent to p components of G-S, we say it sends p edges to the components of G-S; if a component C is adjacent to q vertices in S, we say C sends q edges to S. Then the vertices in S send totally $k_1 + k_2 + \ldots + k_s$ edges to the components of G-S, whereas the components of G-S send at least $(k_{m_1} + k) + (k_{m_2} + k) + \ldots + (k_{m_t} + k)$ edges to S. Hence we have

$$k_1 + k_2 + \ldots + k_s \geq (k_{m_1} + k) + (k_{m_2} + k) + \ldots + (k_{m_t} + k) \tag{1}$$

So

$$\sum_{i=1}^{t} k_i + \sum_{j=t+1}^{s} k_j \geq \sum_{i=1}^{t} k_{m_i} + tk. \tag{2}$$

Claim 2: $\sum_{i=1}^{t} k_i \leq \sum_{i=1}^{t} k_{m_i}$.

We shall prove $k_{m_i} \geq k_i$ (i = 1,2,...,t) by induction, and then Claim 2 follows. By the definition of k_{m_i}, we have $k_{m_1} \geq k_1$.

Assume $k_{m_i} \geq k_i$ for all i such that $i < j$. Now assume i = j. If there is a component $C_p \in \{C_1, C_2, ..., C_j\}$ such that C_p is adjacent to v_q for $q \geq j$, then $k_{m_j} \geq k_{m_p} \geq k_q \geq k_j$. Otherwise, $C_1, C_2, ..., C_j$ are adjacent only to $v_1, v_2, ..., v_{j-1}$. Then $k_1 + k_2 + ... + k_{j-1} \geq (k_{m_1} + k) + (k_{m_2} + k) + ... + (k_{m_j} + k)$. By the induction assumption, $k_{m_i} \geq k_i$ (i = 1,2,...,j-1), and $k_{m_j} \geq 1$. So $k_{m_1} + k_{m_2} + ... + k_{m_{j-1}} + k_{m_j} > k_1 + k_2 + ... + k_{j-1}$, which contradicts the above inequality. Hence $k_{m_j} \geq k_j$.

By (2) and Claim 2, we have

$$\sum_{j=t+1}^{s} k_j \geq tk. \tag{3}$$

But at most t components are adjacent to each of $v_1, v_2, ..., v_s$, then

$$k_i \leq t \quad (i = 1, 2, ..., s). \tag{4}$$

By (3) and (4),

$$(s-t)t \geq \sum_{j=t+1}^{s} k_i \geq tk. \tag{5}$$

By (5), we have $s - t \geq k$ and then $t \leq s - k$. Hence

$$\omega(G - S) \leq |S| - k. \quad \diamond$$

COROLLARY 2. *Let G be a connected graph with even order. If for each vertex $v \in V(G)$ and for each independent set $S \subseteq N_2(v)$, $|N(v) \cap N(S)| \geq |S| + 2n$, then G is n-extendable.*

PROOF: Suppose G is not n-extendable. Then there are n independent edges $e_i = u_i v_i$ (i = 1,2,...,n) such that $G - \{u_i, v_i | i = 1, 2, ..., n\}$ does not have any perfect matching. Let $G' = G - \{u_i, v_i | i = 1, 2, ..., n\}$. By Tutte's Theorem, there is a set $S' \subseteq V(G')$ such that $o(G' - S') > |S'|$. By parity, $o(G' - S') \geq |S'| + 2$. Let $T = S' \cup \{u_i, v_i | i = 1, 2, ..., n\}$. Then $\omega(G - T) = \omega(G' - S') \geq o(G' - S') \geq |S'| + 2 = |S| - 2n + 2$. But by Theorem 1, $\omega(G - T) \leq |T| - (2n-1) = |T| - 2n + 1$, contradicting the above inequality. $\diamond$

The lower bound of $|N(v) \cap N(S)|$ in Corollary 2 is best possible. Let H = K_{2n} and let u,v $\notin$ V(H). We construct G by joining u and v respectively to every vertex of H. Then G satisfies that for each vertex w and for each independent set $S \subseteq N_2(w)$, $|N(w) \cap N(S)| \geq |S| + 2n - 1$. However G is not n-extendable because there are n independent edges in H which are not contained in a perfect matching of G.

The following result in Corollary 3 was due to Lou [5]. We shall prove that Corollary 2 implies it. In [6], Lou gave counterexamples to show that it does not imply Corollary 2.

COROLLARY 3. *Let G be a connected graph with even order. If for each pair of vertices u and v distance 2 apart, $I(u,v) \geq \alpha^*(u,v) + 2n - 1$, then G is n-extendable.*

PROOF: Suppose G is a graph satisfying the hypotheses of this corollary. We shall prove that G also satisfies the hypotheses of Corollary 2.

Let v be a vertex of G and S be an independent set in $N_2(v)$. Let $S = \{w_1, w_2, ..., w_s\}$ and $T = N(v) \cap N(S) = \{v_1, v_2, ..., v_t\}$. Let $k_i = |\{w_j | v_i w_j \in E(G), w_j \in S\}|$ (i = 1,2,...,t). Without loss of generality, assume $k_1 \leq k_2 \leq ... \leq k_t$. Let $k_{m_j} = \max\{k_i | v_i w_j \in E(G) \text{ and } v_i \in T\}$ (j = 1,2,...,s). Without loss of generality, assume $k_{m_1} \leq k_{m_2} \leq ... \leq k_{m_s}$.

If v_j is adjacent to w_i, as $d(v, w_i) = 2$, by the hypotheses of this corollary, $|N(w_i) \cap N(v)| \geq k_j + 1 + 2n - 1 = k_j + 2n$ and $N(w_i) \cap N(v) \subseteq T$. Considering all vertices in T which are adjacent to w_i, w_i is adjacent to at least $k_{m_i} + 1 + 2n - 1 = k_{m_i} + 2n$ vertices in T. The vertices in T send totally $k_1 + k_2 + ... + k_t$ edges to S, and the vertices in S send at least $(k_{m_1} + 2n) + (k_{m_2} + 2n) + ... + (k_{m_s} + 2n)$ edges to T. So

$$k_1 + k_2 + ... + k_t \geq (k_{m_1} + 2n) + (k_{m_2} + 2n) + ... + (k_{m_s} + 2n) \qquad (1)$$

By (1),

$$\sum_{i=1}^{s} k_i + \sum_{j=s+1}^{t} k_j \geq \sum_{j=1}^{s} k_{m_j} + 2ns. \qquad (2)$$

In the following, we shall prove by induction that

$$k_i \leq k_{m_i} \quad (i = 1, 2, ..., s). \qquad (3)$$

By the definition of k_{m_i}, we have $k_{m_1} \geq k_1$. Assume $k_{m_i} \geq k_i$ for all i such that $i < j$. Now assume i = j. If there is a vertex $w_p \in \{w_1, w_2, ..., w_j\}$ such that $w_p v_q \in E(G)$ for $q \geq j$, then $k_{m_j} \geq k_{m_p} \geq k_q \geq k_j$. Otherwise $N(\{w_1, w_2, ..., w_j\}) \cap T \subseteq \{v_1, v_2, ..., v_{j-1}\}$. Then $k_1 + k_2 + ... + k_{j-1} \geq (k_{m_1} + 2n) + (k_{m_2} + 2n) + ... + (k_{m_j} + 2n)$. However, by the induction hypothesis, $k_{m_i} \geq k_i$ (i=1,2,...,j-1), and $k_{m_j} \geq 1$, hence $k_{m_1} + k_{m_2} + ... + k_{m_{j-1}} + k_{m_j} > k_1 + k_2 + ... + k_{j-1}$, contradicting the above inequality.

By (2) and (3), we have

$$\sum_{j=s+1}^{t} k_j \geq 2ns. \qquad (4)$$

But by the definition of k_i,

$$k_i \leq s \quad (i = 1, 2, ..., t). \qquad (5)$$

So

$$(t-s)s \geq \sum_{j=s+1}^{t} k_j \geq 2ns. \qquad (6)$$

By (6), $t - s \geq 2n$. So $t \geq s + 2n$. And hence

$$|N(v) \cap N(S)| \geq |S| + 2n. \quad \diamond$$

In the following we shall prove that Corollary 2 also implies a sufficient condition set up by Plummer [7].

COROLLARY 4. *Let G be a connected graph with even order. If $\delta(G) \geq \nu/2+n$, then G is n-extendable.*

PROOF: By [5], we know that Corollary 3 implies Corollary 4. $\diamond$

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