The number of digraphs with small diameter *

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Abstract

Let $D(n; d = k)$ denote the number of digraphs of order n and diameter equal to k. In this paper it is proved that:

i) for every fixed $k \geq 3$,

$$D(n; d = k) = 4^{\binom{n}{2}}(3 \cdot 2^{-k+1} + o(1))^n;$$

ii) for every fixed $k \geq 1$,

$$n!\, 2^{\binom{n}{2}} S_{k-1}(n) \leq D(n; d = n - k) \leq n!\, 2^{\binom{n}{2}} R_{k-1}(n),$$

where $R_{k-1}(n)$ and $S_{k-1}(n)$ are polynomials of degree $k - 1$ in n with positive leading coefficients depending only on k.

This extends the corresponding results for undirected graphs given in [2].

1 Notation and preliminary results

For a digraph G the outdegree $d^+(x)$ of a vertex x is the number of vertices of G that are adjacent from x and the indegree $d^-(x)$ is the number of vertices of G adjacent to x. For a strongly connected digraph G the distance $d(x, y)$ from vertex x to vertex y is the length of a shortest path of the form $(x, \ldots, y)$. The eccentricity of a vertex x is $ecc(x) = max_{y \in V(G)} d(x, y)$. The diameter of G, denoted $d(G)$ is equal to $max_{x,y \in V(G)} d(x, y)$ if G is strongly connected and ∞ otherwise.

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Now suppose that $V(G) = \{1, \ldots, n\}$ and denote by $A_{ij}^{(k)}$ the set of digraphs with vertex set $\{1, \ldots, n\}$ such that $d(i,j) \geq k$. By $D(n; d = k)$ and $D(n; d \geq k)$ we denote the number of digraphs G of order n and diameter $d(G) = k$ and $d(G) \geq k$, respectively.

Using the material given in Chapter VII, p. 131 of the book by Bollobás [1], it is routine to show that almost all digraphs have diameter two.

Let

$$f(n; n_1, \ldots, n_k) = \binom{n}{n_1, \ldots, n_k} 2^{\sum_{i=1}^{k} \binom{n_i}{2}} \prod_{i=1}^{k-1} (2^{n_i} - 1)^{n_{i+1}}$$

where $n_1 + \ldots + n_k = n$ and $n_i \geq 1$ for every $1 \leq i \leq k$ and

$$f(n,k) = max_{\substack{n_1+\ldots+n_k=n \\ n_1,\ldots,n_k \geq 1}} f(n; n_1, \ldots, n_k).$$

This arithmetical function is the key for obtaining an asymptotic formula for the number of digraphs of diameter k and order n as k is fixed and $n \to \infty$. Its asymptotic behavior was deduced in [2] and is stated in Lemma 1.1.

Lemma 1.1 *For every $k \geq 3$ we have*

$$f(n,k) = 2^{\binom{n}{2}} (3 \cdot 2^{-k+2} + o(1))^n.$$

The following lemmas will be useful in the proofs of the theorems given in the next section.

Lemma 1.2 *The number of bipartite digraphs G whose partite sets are A, B ($A \cap B = \emptyset$, $|A| = p$, $|B| = q$) such that $d^-(x) \geq 1$ for every $x \in B$ and all edges are directed from A towards B is equal to $(2^p - 1)^q$.*

Proof: Since each vertex in B must have at least one incoming edge from some vertex in A, there are $2^p - 1$ choices for the set of incoming edges to any vertex in B. Thus there are $(2^p - 1)^q$ choices for the incoming edges to the set of q vertices in B. □

Lemma 1.3 *The following equality holds:*

$$lim_{n\to\infty} \frac{D(n; d = 3)}{D(n; d \geq 4)} = \infty.$$

Proof: A straightforward computation leads to

$$|A_{ij}^{(3)}| = 2 \cdot 12^{n-2} \cdot 4^{\binom{n-2}{2}} = 3^{n-2} \cdot 2^{\binom{n}{2} + \binom{n-2}{2}}$$

for every $1 \leq i,j \leq n$ and $i \neq j$. Indeed, since $d(i,j) \geq 3$ we deduce that $(i,j) \notin E(G)$ and for every vertex $k \neq i,j$, if $(i,k) \in E(G)$ then $(k,j) \notin E(G)$. This implies that for every fixed choice of the subdigraph induced by $\{i,j\}$ (and this can be done in exactly two ways), then for every $k \neq i,j$ the subdigraph induced by $\{i,j,k\}$ can be chosen in 12 ways. Since the subdigraph induced by $n-2$ vertices different from i and j can be chosen in $4^{\binom{n-2}{2}}$ ways, the formula follows.

The number of digraphs in $A_{ij}^{(4)}$ such that $d^+(i) = n_1$ and $d^-(j) = n_2$ is equal to

$$\begin{aligned} & 4^{\binom{n_1}{2}+\binom{n_2}{2}+\binom{n-2-n_1-n_2}{2}+(n_1+n_2)(n-2-n_1-n_2)} \cdot 2^{n_1+n_2+n_1n_2+1+2(n-2-n_1-n_2)+n_1+n_2} \\ = {} & 2^{\binom{n}{2}+\binom{n-2}{2}-n_1n_2}. \end{aligned}$$

To justify this formula let $X = \{x \mid (i,x) \in E(G)\}$ and $Y = \{y \mid (y,j) \in E(G)\}$; it follows that $|X| = n_1$ and $|Y| = n_2$. Now $d(i,j) \geq 4$ implies that $X \cap Y = \emptyset$ and the directed edges between: a) vertices in X; b) vertices in Y; c) vertices in $\{1,\ldots,n\} \backslash (X \cup Y \cup \{i,j\})$; d) vertices in $X \cup Y$ in a part and vertices in $\{1,\ldots,n\} \backslash (X \cup Y \cup \{i,j\})$ in another part, can be chosen in

$$4^{\binom{n_1}{2}+\binom{n_2}{2}+\binom{n-2-n_1-n_2}{2}+(n_1+n_2)(n-2-n_1-n_2)}$$

ways. Also the directed edges from: e) X to i; f) j to Y; g) Y to X; h) j to i; i) $\{1,\ldots,n\} \backslash (X \cup Y \cup \{i,j\})$ to i; j) j to $\{1,\ldots,n\} \backslash (X \cup Y \cup \{i,j\})$; k) j to X and l) Y to i, can be chosen in $2^{n_1+n_2+n_1n_2+1+2(n-2-n_1-n_2)+n_1+n_2}$ ways.

It follows that for every $1 \leq i < j \leq n$ we have

$$\begin{aligned} |A_{ij}^{(4)}|/2^{\binom{n-2}{2}+\binom{n}{2}} &= \sum_{\substack{n_1+n_2+n_3=n-2 \\ n_1,n_2,n_3 \geq 0}} \binom{n-2}{n_1, n_2, n_3} 2^{-n_1n_2} \\ &= \sum_{k=0}^{n-2} \sum_{\substack{n_2+n_3=n-2-k \\ n_2,n_3 \geq 0}} \binom{n-2}{k, n_2, n_3} 2^{-kn_2} \\ &= \sum_{k=0}^{n-2} \binom{n-2}{k} \sum_{n_2=0}^{n-2-k} \binom{n-2-k}{n_2} 2^{-kn_2} \\ &= \sum_{k=0}^{n-2} \binom{n-2}{k} (1+2^{-k})^{n-2-k}. \end{aligned}$$

We have $|A_{ij}^{(4)}| < 2^{\binom{n}{2}+\binom{n-2}{2}}(2^{n-2} + (\frac{5}{2})^{n-2})$ because $2^{-k} \leq \frac{1}{2}$ for every $k \geq 1$. We can write

$$\begin{aligned} D(n; d=3) &= D(n; d \geq 3) - D(n; d \geq 4); \\ D(n; d \geq 3) &= |\bigcup_{\substack{1 \leq i,j \leq n \\ i \neq j}} A_{ij}^{(3)}| > 3^{n-2} \cdot 2^{\binom{n}{2}+\binom{n-2}{2}}; \\ D(n; d \geq 4) &= |\bigcup_{\substack{1 \leq i,j \leq n \\ i \neq j}} A_{ij}^{(4)}| < \sum_{\substack{1 \leq i,j \leq n \\ i \neq j}} |A_{ij}^{(4)}| \\ &< (n^2-n) 2^{\binom{n}{2}+\binom{n-2}{2}} (2^{n-2} + (\frac{5}{2})^{n-2}) \end{aligned}$$

and the proof follows. □

2 Main results

Theorem 2.1 *For every fixed $k \geq 3$ we have*

$$D(n; d = k) = 4^{\binom{n}{2}}(3 \cdot 2^{-k+1} + o(1))^n.$$

Proof: If $k = 3$ we have $D(n; d = 3) \sim D(n; d \geq 3)$ by Lemma 1.3 and also $D(n; d \geq 3) = |\bigcup_{1 \leq i,j \leq n, i \neq j} A_{i,j}^{(3)}| = 4^{\binom{n}{2}}(\frac{3}{4} + o(1))^n$ since $|A_{ij}^{(3)}| = 3^{n-2} \cdot 2^{\binom{n}{2}+\binom{n-2}{2}}$ for every $i \neq j$ and $|A_{ij}^{(3)}| \leq |\bigcup_{1 \leq i,j \leq n, i \neq j} A_{ij}^{(3)}| \leq (n^2 - n)|A_{ij}^{(3)}|$.

Let $k \geq 4$. If $x \in V(G)$ has $ecc(x) = k$, then

$$V_1(x) \cup \ldots \cup V_k(x)$$

is a partition of $V(G) \backslash \{x\}$, where $V_i(x) = \{y \mid y \in V(G) \text{ and } d(x, y) = i\}$ for $0 \leq i \leq k$. It follows that there are directed edges from x towards all vertices of $V_1(x)$. Furthermore, for every $2 \leq i \leq k$ and any vertex $z \in V_i(x)$ there exists a directed edge (t, z), where $t \in V_{i-1}(x)$. Let n_i be the number of vertices in $V_i(x), 1 \leq i \leq k$. By Lemma 1.2 we get

$$\begin{aligned} &|\{G \mid V(G) = \{1, \ldots, n\} \text{ and } ecc(x) = k\}| \\ = &\sum_{\substack{n_1 + \ldots + n_k = n-1 \\ n_1, \ldots, n_k \geq 1}} \binom{n-1}{n_1, \ldots, n_k} 4^{\sum_{i=1}^k \binom{n_i}{2}} \prod_{i=1}^{k-1} (2^{n_i} - 1)^{n_{i+1}} \prod_{i=1}^{k} 2^{n_i(n_{i-1} + \ldots + 1)} \\ = &\; 2^{\binom{n}{2}} \sum_{\substack{n_1 + \ldots + n_k = n-1 \\ n_1, \ldots, n_k \geq 1}} f(n-1; n_1, \ldots, n_k) \end{aligned}$$

because

$$2^{\sum_{i=1}^k \binom{n_i}{2}} \prod_{i=1}^{k} 2^{n_i(n_{i-1} + \ldots + 1)} = 2^{\binom{n}{2}}.$$

One obtains

$$\sum_{\substack{n_1 + \ldots + n_k \\ n_1, \ldots, n_k \geq 1}} f(n-1; n_1, \ldots, n_k) \leq \binom{n-2}{k-1} f(n-1, k)$$

since the number of compositions $n - 1 = n_1 + \ldots + n_k$ having k positive terms equals $\binom{n-2}{k-1}$. This implies that

$$|\{G \mid V(G) = \{1, \ldots, n\}, ecc(x) = k\}| \leq 2^{\binom{n}{2}} \binom{n-2}{k-1} f(n-1, k).$$

Hence

$$\begin{aligned} D(n; d = k) &\leq |\bigcup_{x \in \{1, \ldots, n\}} \{G \mid V(G) = \{1, \ldots, n\} \text{ and } ecc(x) = k\}| \\ &\leq n 2^{\binom{n}{2}} \binom{n-2}{k-1} f(n-1, k) = 4^{\binom{n}{2}}(3 \cdot 2^{-k+1} + o(1))^n \end{aligned}$$

by Lemma 1.1.

In order to show the opposite inequality we shall generate a large class of digraphs of order n and diameter equal to k as follows:
Let $x \in \{1, ..., n\}$ be a fixed vertex. We consider the class of digraphs G such that:

i) $ecc(x) = k$;

ii) $|V_1(x)| = |V_2(x)| = \ldots = |V_{r-1}(x)| = 1$; $|V_r(x)| = \alpha(n,k) = \lfloor (n-k+1)/3 \rfloor$; $|V_{r+1}(x)| = \beta(n,k) = \lceil 2(n-k+1)/3 \rceil$; $|V_{r+2}(x)| = |V_{r+3}(x)| = \ldots = |V_k(x)| = 1$ for odd k, where $r = (k-1)/2$, and $|1(x)| = |V_2(x)| \ldots = |V_r(x)| = 1$; $|V_{r+1}(x)| = \alpha(n,k)$; $|V_{r+2}(x)| = \beta(n,k)$; $|V_{r+3}(x)| = |V_{r+4}(x)| = \ldots = |V_k(x)| = 1$ for even k, where $r = k/2 - 1$, respectively;

iii) classes $V_r(x)$ and $V_{r+1}(x)$ for odd k and $V_{r+1}(x)$ and $V_{r+2}(x)$ for even k, respectively induce digraphs of diameter equal to 2;

iv) $(c,x), (c,a), (c,b) \in E(G)$, where $V_1(x) = \{a\}, V_2(x) = \{b\}$ and $V_k(x) = \{c\}$.

If G denotes a digraph produced by this procedure it is easy to see that $|V(G)| = n, ecc(x) = k$ and $d(G) = k$.

Since almost all digraphs of order n have diameter equal to two as $n \to \infty$, it follows that the number of digraphs generated in this way is asymptotically equal to

$$\frac{1}{8} 2^{\binom{n}{2}} f(n-1; 1, \ldots, 1, \alpha(n,k), \beta(n,k), 1, \ldots, 1).$$

By denoting $\alpha = \alpha(n,k) = \frac{n-k+1}{3} - \epsilon$; $\beta = \beta(n,k) = \frac{2n-2k+2}{3} + \epsilon$, we get

$$\begin{aligned} f(n-1; 1, \ldots, 1, \alpha, \beta, 1, \ldots, 1) &= \frac{(n-1)!}{\alpha! \beta!} 2^{\binom{\alpha}{2} + \binom{\beta}{2}} (2^\alpha - 1)^\beta (2^\beta - 1) \\ &\sim \frac{(n-1)!}{\alpha! \beta!} 2^{\frac{1}{2}((\alpha+\beta)^2 - \alpha + \beta)}. \end{aligned}$$

By Stirling's formula we find that $\frac{(n-1)!}{\alpha!\beta!} \sim P_k(n) n^{1/2} 3^n \cdot 2^{-2n/3}$, where $P_k(n)$ is a polynomial in n of fixed degree (depending only on k) and

$$2^{\frac{1}{2}((\alpha+\beta)^2 - \alpha + \beta)} = C \cdot 2^{n^2/2 - kn + 7n/6}$$

where $C > 0$ is a constant. Hence this number of digraphs is asymptotically equal to

$$C \cdot 2^{\binom{n}{2} - 3} P_k(n) n^{1/2} 3^n \cdot 2^{\binom{n}{2} - kn + n} \sim 4^{\binom{n}{2}} (3 \cdot 2^{-k+1} + o(1))^n.$$

□

Theorem 2.2 *The following inequalities*

$$n!\, 2^{\binom{n}{2}} S_{k-1}(n) \leq D(n; d = n-k) \leq n!\, 2^{\binom{n}{2}} R_{k-1}(n)$$

hold for every fixed $k \geq 1$, where $R_{k-1}(n)$ and $S_{k-1}(n)$ are polynomials of degree $k-1$ in n with positive leading coefficients depending only on k.

Proof: If $n_1+\ldots+n_{n-k}=n-1$, k is fixed and as $n\to\infty$ almost all $n_1,\ldots,n_{n-k}$ are equal to 1 then the corresponding factors $(2^{n_i}-1)^{n_{i+1}}=1$ for $n_i=1$ in the expression $f(n-1;n_1,\ldots,n_{n-k})$. Since $D(n;d=k)\le n\,2^{\binom{n}{2}}\binom{n-2}{k-1}f(n-1,k)$ it follows that $D(n;d=n-k)\le n!\,2^{\binom{n}{2}}\binom{n-2}{k-1}C_1(k)$, where $C_1(k)$ is a constant depending only on k. Indeed, in the composition $n-1=n_1+\ldots+n_{n-k}$ where $n_i\ge 1$ at most k terms are greater than 1 and any of them is less than or equal to $k+1$. Hence $f(n-1,n-k)\le(n-1)!2^{k\binom{k+1}{2}}(2^{k+1}-1)^{k(k+1)}$. Therefore $D(n;d=n-k)\le n!\,2^{\binom{n}{2}}R_{k-1}(n)$, where $R_{k-1}(n)$ is a polynomial of degree $k-1$ in n with positive leading coefficient depending only on k.

In order to prove the other inequality we shall generate a large class $\mathcal{C}$ of digraphs of order n and diameter $n-k$ as follows:
For every subset $X\subset\{1,\ldots,n\}$ of cardinality $|X|=n-k+1$ we consider a Hamiltonian directed path $(x_1,\ldots,x_{n-k+1})$ on vertex set X. The remaining $k-1$ vertices y will be joined each by directed edges in both directions (y,x) and (x,y) with the vertices x in the set $\{x_3,x_4,\ldots,x_{n-k-1}\}$ in $(n-k-3)^{k-1}$ ways.

All digraphs in $\mathcal{C}$ contain directed edges (x_{n-k+1},x_1) and (x_{n-k+1},x_2). Any two vertices in $\{1,\ldots,n\}\backslash X$ are not adjacent in any direction and now the backward directed edges (u,v) where $u\in V_j(x_1)$ and $v\in V_i(x_1)$ such that $0\le i<j\le n-k$ can be drawn in

$$2^{\binom{n}{2}-\binom{k-1}{2}-(k-1)-2}$$

ways. It is easy to see that each digraph produced in this way has diameter $n-k$. We shall prove that all digraphs generated by this procedure are pairwise distinct. Indeed, for a fixed Hamiltonian path $(x_1,\ldots,x_{n-k+1})$ all digraphs produced are pairwise distinct since all partitions $V_1(x_1)\cup\ldots\cup V_{n-k}(x_1)$ of $\{1,\ldots,n\}\backslash\{x_1\}$ generated by this algorithm are pairwise distinct. Note that if a vertex $y\in\{1,\ldots,n\}\backslash X$ and a vertex $x_i\in X$ appear in the same class $V_j(x_1)$ they do not have a symmetric role since $(x_i,x_{i+1})\in E(G)$ but $(y,x_{i+1})\notin E(G)$ for any digraph $G\in\mathcal{C}$.

Now suppose that a digraph G_1 built by starting from a Hamiltonian path $(x_1,\ldots,x_{n-k+1})$ coincides with a digraph G_2 built from a Hamiltonian path $(z_1,\ldots,z_{n-k+1})$, where $(z_1,\ldots,z_{n-k+1})\ne(x_1,\ldots,x_{n-k+1})$ are distinct permutations of the set $\{x_1,\ldots,x_{n-k+1}\}$. We shall consider separately two subcases: the first for $x_1\ne z_1$ and the second for $x_1=z_1$.

Case 1: Since $x_1\ne z_1$ there exists $i\ge 2$ such that $x_1=z_i$. Because $(x_1,x_2),(x_2,x_3),\ldots,(x_{n-k},x_{n-k+1}),(x_{n-k+1},x_1),(x_{n-k+1},x_2)\in E(G_1)$ and $(z_i,z_j)\notin E(G_1)$, where $1\le i<j\le n-k+1$ and $j\ge i+2$, it follows that $z_{i+1}=x_2,z_{i+2}=x_3,\ldots,z_{n-k+1}=x_{n-k-i+2},\ldots,z_s=x_{n-k+1}$, where $s<i$. We deduce that $(x_{n-k+1},x_2)=(z_s,z_{i+1})\in E(G_2)$ where $s\le i-1$, a contradiction.

Case 2: If $x_1=z_1$ it follows that $z_2=x_2,\ldots,z_{n-k+1}=x_{n-k+1}$ which contradicts the hypothesis.

Since all digraphs generated in this way are pairwise distinct it follows that

$$|\mathcal{C}|=\binom{n}{k-1}(n-k+1)!\,(n-k-3)^{k-1}2^{\binom{n}{2}-\binom{k-1}{2}-k-1}=2^{\binom{n}{2}}n!\,S_{k-1}(n),$$

therefore $D(n; d = n-k) \geq n!\, 2^{\binom{n}{2}} S_{k-1}(n)$, where $S_{k-1}(n)$ is a polynomial of degree $k-1$ in n with positive leading coefficient depending only on k. □

Corollary 2.3 *For every fixed $k \geq 2$ the following equalities hold:*

$$lim_{n\to\infty} \frac{D(n; d = k)}{D(n; d = k+1)} = lim_{n\to\infty} \frac{D(n; d = n-k)}{D(n; d = n-k+1)} = \infty.$$

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References

[1] B. Bollobás. *Graph Theory. An introductory course*, Springer-Verlag, New York Heidelberg Berlin, 1979.

[2] I. Tomescu. An asymptotic formula for the number of graphs having small diameter, *Discrete Mathematics* (to be published).

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