# Further results on the existence of HSOLSSOM($h^n$)


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**Abstract** In this paper, we improve the existence results for holey self-orthogonal Latin squares with symmetric orthogonal mates (HSOLSSOMs), especially for type $6^n$. We are also able to construct three new unipotent SOLSSOMs of orders 46, 54 and 58, the existence of which is previously unknown.


## 1. Introduction

A *quasigroup* is an ordered pair $(Q, \cdot)$, where Q is a set and $(\cdot)$ is a binary operation on Q such that the equations

$$a \cdot x = b \quad \text{and} \quad y \cdot a = b$$

are uniquely solvable for every pair of elements a, b in Q. It is well known (e.g., see [6]) that the multiplication table of a quasigroup defines a *Latin square*; that is , a Latin square can be viewed as the multiplication table of a quasigroup with the headline and sideline removed. For a finite set Q, the *order* of the quasigroup $(Q, \cdot)$ is $|Q|$. A quasigroup $(Q, \cdot)$ is called *idempotent* if the identity

$$x^2 = x$$

holds for all x in Q.

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Two quasigroups $(Q, \cdot)$ and $(Q, *)$ defined on the same set Q are said to be *orthogonal* if the pair of equations $x \cdot y = a$ and $x * y = b$, where a and b are any two given elements of Q, are satisfied simultaneously by a unique pair of elements from Q. We remark that when two quasigroups are orthogonal, then their corresponding Latin squares are also orthogonal in the usual sense.

Let S be a set and $\boldsymbol{H} = \{S_1, S_2, ..., S_n\}$ be a set of subsets of S. A *holey Latin square* having *hole set* $\boldsymbol{H}$ is an $|S| \times |S|$ array L, indexed by S, satisfying the following properties:

(1) every cell of L either contains an element of S or is empty,

(2) every element of S occurs at most once in any row or column of L,

(3) the subarrays indexed by $S_i \times S_i$ are empty for $1 \le i \le n$ (these subarrays are referred to as *holes*),

(4) element $s \in S$ occurs in row or column t if and only if $(s, t) \in (S \times S) \setminus \cup_{1 \le i \le n}(S_i \times S_i)$.

The *order* of L is $|S|$. Two holey Latin squares on symbol set S and hole set $\boldsymbol{H}$, say $L_1$ and $L_2$, are said to be *orthogonal* if their superposition yields every ordered pair in $(S \times S) \setminus \cup_{1 \le i \le n}(S_i \times S_i)$. We shall use the notation IMOLS(s; $s_1$, ..., $s_n$) to denote a pair of orthogonal holey Latin squares on symbol set S and hole set $\boldsymbol{H} = \{S_1, S_2, ..., S_n\}$, where $s = |S|$ and $s_i = |S_i|$ for $1 \le i \le n$. If $\boldsymbol{H} = \varnothing$, we obtain a MOLS(s). If $\boldsymbol{H} = \{S_1\}$, we simply write IMOLS(s, $s_1$) for the orthogonal pair of holey Latin squares.

If $\boldsymbol{H} = \{S_1, S_2, ..., S_n\}$ is a partition of S, then a holey Latin square is called a *partitioned incomplete Latin square,* denoted by PILS. The *type* of the PILS is defined to be the multiset $\{|S_i|: 1 \le i \le n\}$. We shall use an "exponential" notation to describe types: so type $t_1^{u_1} \ldots t_k^{u_k}$ denotes $u_i$ occurrences of $t_i$, $1 \le i \le k$, in the multiset. Two orthogonal PILS of type T will be denoted by HMOLS(T).

A holey Latin square is called *self-orthogonal* if it is orthogonal to its transpose. For self-orthogonal holey Latin squares we use the notation SOLS(s), ISOLS(s, $s_1$) and HSOLS(T) for the case of $\boldsymbol{H} = \varnothing$, $\{S_1\}$ and a partition $\{S_1, S_2, ..., S_n\}$, respectively.

If any two PILS in a set of t PILS of type T are orthogonal, then we denote the set by t HMOLS(T). Similarly, we may define t MOLS(s) and t IMOLS(s, $s_1$).

A *holey SOLSSOM* having partition ***P*** is 3 HMOLS (having partition ***P***), say A, B, C, where $B = A^T$ and $C = C^T$. Here a SOLSSOM stands for a *self-orthogonal Latin square* (SOLS) with a *symmetric orthogonal mate* (SOM). A holey SOLSSOM of type T will be denoted by HSOLSSOM(T).

HSOLSSOMs have been useful in the construction of resolvable orthogonal arrays invariant under the Klein 4-group [9], Steiner pentagon systems [11] and three-fold BIBDs with block size seven [18]. The existence of a HSOLSSOM($h^n$) has been investigated by several authors. It is easy to see that $n \geq 5$ is a necessary condition for the existence of such a design. The following existence results are known.

**Theorem 1.1** ([13], [5]) If h is an odd integer, then a HSOLSSOM($h^n$) exists if and only if $n \geq 5$ is odd except possibly for $h = 3$ and $n \in \{11, 15, 19, 23, 27, 39, 51, 59, 87\}$.

**Theorem 1.2** ([12], [17], [5], [2]) If h is an even integer, then a HSOLSSOM($h^n$) exists for all $n \geq 5$ except possibly when

(1) $h \equiv 2 \pmod 4$, $h \neq 6$, and $n \in \{8, 10, 12, 14, 15, 16, 18, 20, 22, 24, 28, 32\}$, and

(2) $h = 6$ and $n \in \{u: u \equiv 0, 2 \pmod 4\} \cup \{u: u \equiv 3 \pmod 4 \text{ and } u \leq 267\}$.

In this paper, we improve the above known results and show the existence of HSOLSSOM($h^n$) when

$h = 3$ and $n \in \{11, 15, 39, 51, 59, 87\}$,

$h \equiv 2 \pmod 4$, $h \neq 6$, and $n \in \{10, 15, 16, 20\}$,

$h = 6$ and $n \in \{u: u \geq 5\} \setminus \{6, 7, 10, 11, 12, 16, 18, 19, 20, 22, 23, 24, 27, 32, 38, 39\}$.

We are also able to construct three new unipotent SOLSSOMs of orders 46, 54 and 58, the existence of which is previously unknown.

## 2. Constructions

Let $K_n$ be the complete undirected graph with n vertices. A *pentagon system* (PS) of order n is a pair ($K_n$, $\boldsymbol{B}$), where $\boldsymbol{B}$ is a collection of edge disjoint pentagons which partition the edges of $K_n$. A *Steiner pentagon system* (SPS) of order n is a pentagon system ($K_n$, $\boldsymbol{B}$) with the additional property that every pair of vertices are joined by a path of length 2 in exactly one pentagon of $\boldsymbol{B}$.

Let Q be an n-set and let $K_n$ be based on Q. It is well known [10] that a quasigroup (Q, ·) satisfying the three identities $x^2 = x$, $(yx)x = y$ and $x(yx) = y(xy)$ is equivalent to a SPS ($K_n$, $\boldsymbol{B}$). Here a pentagon $(x, y, z, u, v) \in \boldsymbol{B}$ if and only if $xy = z$ and $yx = v$ for $x \neq y$ and $x^2 = x$ for all $x \in Q$. A quasigroup associated with a SPS is called a *Steiner pentagon quasigroup* (briefly denoted by SPQ).

A *partitioned incomplete quasigroup* (PIQ) is a partial quasigroup whose multiplication table with the headline and sideline removed is a PILS. The type of the PIQ is the type of its associated PILS. A PIQ of type $h^n$ satisfying the identities $(yx)x = y$ and $x(yx) = y(xy)$ is denoted by HSPQ($h^n$).

A *holey Steiner pentagon system* of type $h^n$ (HSPS($h^n$)) is a SPS with n disjoint holes of equal-size h. A HSPS($h^n$) is essentially equivalent to a HSPQ($h^n$).

**Theorem 2.1** ([2]) Suppose there exists a holey Steiner pentagon system of type $h^n$. Then there exists a HSOLSSOM($h^n$).

We need the following known constructions, which are Lemmas 2.1 and 2.2 in [17].

**Theorem 2.2** Suppose q is an odd prime power, $q \geq 7$. Suppose there exist SOLSSOM(m) and ISOLSSOM($m + e_t$, $e_t$) where m is even, $e_t = 0$ or $e_t$ odd $> 0$, $t = 1, 2, ..., (q-5)/2$, $k = \Sigma_{1 \leq t \leq (q-5)/2}(2e_t)$. Then there exists a HSOLSSOM of type $m^{(q-1)}(m + k)^1$.

**Theorem 2.3** Suppose $q \geq 5$, q is an odd prime power or $q \equiv \pm 1 \pmod 6$. Suppose there exist ISOLSSOM($m + e_t$, $e_t$) where m is even, $e_t = 0$ or $e_t$ odd $> 0$, $t = 1, 2, ..., (q-1)/2$, $k = \Sigma_{1 \leq t \leq (q-1)/2}(2e_t)$. Then there exists a HSOLSSOM of type $m^q k^1$.

We also need several other recursive constructions. The first one is simple but useful.

**Construction 2.4 (Filling in Holes)**

(1) Suppose there exists a HSOLSSOM of type $\{s_i: 1 \le i \le n\}$. Let $a \ge 0$ be an integer. For each i, $1 \le i \le n-1$, if there exists a HSOLSSOM of type $\{s_{ij}: 1 \le j \le k(i)\} \cup \{a\}$, where $s_i = \Sigma_{1 \le j \le k(i)} s_{ij}$, then there is a HSOLSSOM of type $\{s_{ij}: 1 \le j \le k(i), 1 \le i \le n-1\} \cup \{a+s_n\}$.

(2) Suppose there exists a HSOLSSOM of type $\{s_i: 1 \le i \le n\}$. Suppose there exists also a HSOLSSOM of type $\{t_j: 1 \le j \le k\}$, where $s_n = \Sigma_{1 \le j \le k} t_j$. Then there is a HSOLSSOM of type $\{s_i: 1 \le i \le n-1\} \cup \{t_j: 1 \le j \le k\}$.

The next recursive construction for HSOLSSOM uses group divisible designs. A *group divisible design* (or GDD), is a triple ( X, ***G***, ***B*** ) which satisfies the following properties:

(1) ***G*** is a partition of a set X (of *points*) into subsets called *groups*,

(2) ***B*** is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point,

(3) every pair of points from distinct groups occurs in a unique block.

The *group type* of the GDD is the multiset $\{ |G|: G \in \mathcal{G} \}$. A GDD ( X, ***G***, ***B*** ) will be referred to as a K-GDD if $|B| \in K$ for every block B in ***B***. A TD(k, n) is a GDD of group type $n^k$ and block size k. An RTD(k, n) is a TD(k, n) where the blocks can be partitioned into parallel classes. It is well known that the existence of an RTD(k, n) is equivalent to the existence of a TD(k + 1, n) or equivalently k - 1 MOLS(n). We wish to remark that a special GDD with all groups of size one is essentially a pairwise balanced design (PBD), denoted by ( X, ***B*** ). We use [3] as our standard design theory reference .

The following PBD construction is essentially [13, Lemma 3.1].

**Construction 2.5** Suppose there exists a PBD ( X, ***B*** ) and for each block $B \in \mathcal{B}$ there exists a HSOLSSOM($h^{|B|}$). Then there exists a HSOLSSOM($h^{|X|}$).

More generally, we can apply Wilson's fundamental construction for GDDs [15] to obtain a similar construction for HSOLSSOM.

**Construction 2.6 (Weighting)** Suppose (X, *G*, *B*) is a GDD and let $w : X \to \mathbf{Z}^{+} \cup \{0\}$. Suppose there exists a HSOLSSOM of type $\{w(x): x \in B\}$ for every $B \in \mathcal{B}$. Then there exists a HSOLSSOM of type $\{ \sum_{x \in G} w(x) : G \in \mathcal{G} \}$.

The following product construction is essentially Lemma 3.4 in [12].

**Construction 2.7** Suppose there exists a HSOLSSOM of type $h^n$. Let $m \geq 4$ and $m \neq 6, 10$. Then there exists a HSOLSSOM of type $(mh)^n$.

To apply the above constructions the following known results are useful.

**Theorem 2.8** ([6]) For any prime power p, there exists a TD(k, p), where $3 \leq k \leq p + 1$.

**Theorem 2.9** ([4]) (1) $N(q) \geq 4$ if $q \geq 5$ and $q \notin E_4 = \{6, 10, 14, 18, 22\}$.

(2) $N(q) \geq 5$ if $q \geq 7$ and $q \notin E_5 = E_4 \cup \{15, 20, 26, 30, 34, 38, 39, 46, 54, 60, 62\}$.

## 3. New results for HSOLSSOM($h^n$)

In this section, we shall improve the known results in Theorems 1.1 and 1.2.

**Lemma 3.1** There exists a HSOLSSOM of type $3^n$ for $n \in \{11, 15, 39, 51, 59, 87\}$.

Proof: A HSPS($3^{11}$) is shown in [1]. Then a HSOLSSOM of type $3^{11}$ follows. A 7-GDD of type $3^{15}$ is known (see [4] for example). Apply Weighting Construction and give weight one to each point of the GDD. This takes care of the case n = 15.

Start with an RTD(9, 9) and delete 8 points in the last three groups. We get a {6, 7, 9}-GDD of type $6^8 9^1$. Give weight two to each point and use HSOLSSOMs of types $2^6$, $2^7$, and $2^9$ as input designs. We get a HSOLSSOM of type $12^8 18^1$. Add three new points and fill in holes with HSOLSSOMs of types $3^5$ and $3^7$ to get a HSOLSSOM of type $3^{39}$.

Start with a HSOLSSOM of type $30^5$. Add three new points and fill in holes with a HSOLSSOM of type $3^{11}$ to get a HSOLSSOM of type $3^{51}$.

Delete three points from a group of a TD(6, 15) and give weight two to each point to get a HSOLSSOM of type $30^5 24^1$. Add three new points and fill in holes with HSOLSSOMs of types $3^9$ and $3^{11}$ to get a HSOLSSOM of type $3^{59}$.

Delete four points from a group of an RTD(7, 19). We get a {6, 7}-GDD of type $6^{19}15^1$. Give weight two to each point and use HSOLSSOMs of types $2^6$ and $2^7$ as input designs. We get a HSOLSSOM of type $12^{19}30^1$. Add three new points and fill in holes with HSOLSSOMs of types $3^5$ and $3^{11}$ to get a HSOLSSOM of type $3^{87}$. □

**Lemma 3.2** There exists a HSOLSSOM($h^n$) for $h \equiv 2 \pmod 4$, $h \neq 6$, and $n \in \{10, 15, 16, 20\}$.

Proof: A HSPS of type $2^n$ comes from [1]. Then a HSOLSSOM of the same type also exists. The conclusion follows from Construction 2.7. □

In what follows, we shall deal with the type $6^n$. We need some working lemmas.

**Lemma 3.3** Suppose there exists a TD(k + 2, 3m). Suppose d, a and b are integers, $d \in \{0, 1\}$ and $a, b \leq m$. Suppose there exist HSOLSSOMs of types $2^k$, $2^{k+1}$ and $2^{k+2}$. If there exist HSOLSSOM($6^{u+d}$) for u = m, a and b, then there exists a HSOLSSOM($6^n$) for $n = km + a + b + d$.

Proof: Delete some points from the last two groups of the TD, leaving 3a and 3b points respectively. Give weight two to each point and apply the Weighting Construction. Since the input HSOLSSOMs all exist, we get a HSOLSSOM of type $(6m)^k(6a)^1(6b)^1$. Add 6d new points and fill in the holes with known HSOLSSOMs of types $6^{u+d}$, u = m, a and b. This gives the desired HSOLSSOM. □

**Lemma 3.4** Suppose t, a and b are integers, t = 2 or $t \geq 4$, $a, b \leq 4t + 1$. If there exist HSOLSSOM($6^n$) for n = a and b, then there exists a HSOLSSOM($6^n$) for $n = 5(4t + 1) + a + b$.

Proof: Apply Lemma 3.3 with m = 4t + 1, k = 5 and d = 0. From Theorem 2.9, we have a TD(7, 3(4t + 1)). The required HSOLSSOM($6^{4t+1}$) comes from Theorem 1.2. □

**Lemma 3.5** Suppose there exists a TD(k + 1, 3m). Suppose d and a are integers, $d \in \{0, 1\}$ and $a \leq m$. Suppose there exist HSOLSSOMs of types $2^k$ and $2^{k+1}$. If there exist HSOLSSOM($6^{u+d}$) for u = m and a, then there exists a HSOLSSOM($6^n$) for $n = km + a + d$.

Proof: Delete some points from the last group of the TD, leaving 3a points. Give weight two to each point and apply the Weighting Construction. Since the input HSOLSSOMs all exist, we get a HSOLSSOM of type $(6m)^k(6a)^1$. Add 6d new points and fill in the holes with known HSOLSSOMs of types $6^{u+d}$, u = m and a. This gives the desired HSOLSSOM. □

**Lemma 3.6** Suppose t and a are integers, $t \geq 1$, $a \leq 4t + 1$. If there exists a HSOLSSOM($6^a$), then there exists a HSOLSSOM($6^n$) for $n = 5(4t + 1) + a$.

Proof: Apply Lemma 3.5 with $m = 4t + 1$, $k = 5$ and $d = 0$. From Theorem 2.9, we have a TD(6, 3(4t + 1)). The required HSOLSSOM($6^{4t+1}$) comes from Theorem 1.2. □

**Lemma 3.7** If $n_0 \in \{26, 30, 54, 78, 102; 87, 91, 95, 99, 103; 108, 112, 116, 100, 104\}$, then there is a HSOLSSOM($6^n$) whenever $n \equiv n_0 \pmod{20}$ and $n \geq n_0$.

Proof: Apply Lemmas 3.4 and 3.6 with the parameters shown in Table 3.1. The required input HSOLSSOMs of types $6^n$ for n = 8, 14 and 15 can be done as follows. Deleting one point from a TD(7, 7) gives a 7-GDD of type $6^8$. Give each point weight one. This solves the first case. Start with a known 7-GDD of type $3^{15}$. Deleting one group gives a {6, 7}-GDD of type $3^{14}$. Give weight two to each point of the two GDDs. We get the last two cases. □

**Lemma 3.8** There exists a HSOLSSOM($6^n$) for $n \equiv 2 \pmod 4$, $n \geq 5$ and $n \notin \{6, 10, 18, 22, 38\}$.

Proof: From Lemma 3.7, we need only deal with the cases n = 34, 42, 58, 62 and 82. For n = 34, we add three new points to three parallel classes of a 9-RGDD of type $3^{33}$ (see [4] for its existence) to get a {9, 10}-GDD of type $3^{34}$. Giving weight two to each point solves this case. For the remaining cases, apply Lemma 3.3 with d = 0 and other parameters shown in Table 3.2. □

| t ≥ | a | b | n = 5(4t+1)+a+b | n ≥ | Authority |
|---|---|---|---|---|---|
| 1 | 1 | 0 | n ≡ 6 (mod 20) | 26 | Lemma 3.6 |
| 1 | 5 | 0 | n ≡ 10 (mod 20) | 30 | Lemma 3.6 |
| 2 | 9 | 0 | n ≡ 14 (mod 20) | 54 | Lemma 3.6 |
| 3 | 13 | 0 | n ≡ 18 (mod 20) | 78 | Lemma 3.6 |
| 4 | 17 | 0 | n ≡ 2 (mod 20) | 102 | Lemma 3.6 |
| 4 | 1 | 1 | n ≡ 7 (mod 20) | 87 | Lemma 3.4 |
| 4 | 5 | 1 | n ≡ 11 (mod 20) | 91 | Lemma 3.4 |
| 4 | 5 | 5 | n ≡ 15 (mod 20) | 95 | Lemma 3.4 |
| 4 | 9 | 5 | n ≡ 19 (mod 20) | 99 | Lemma 3.4 |
| 4 | 9 | 9 | n ≡ 3 (mod 20) | 103 | Lemma 3.4 |
| 4 | 15 | 8 | n ≡ 8 (mod 20) | 108 | Lemma 3.4 |
| 4 | 14 | 13 | n ≡ 12 (mod 20) | 112 | Lemma 3.4 |
| 4 | 17 | 14 | n ≡ 16 (mod 20) | 116 | Lemma 3.4 |
| 4 | 14 | 1 | n ≡ 0 (mod 20) | 100 | Lemma 3.4 |
| 4 | 14 | 5 | n ≡ 4 (mod 20) | 104 | Lemma 3.4 |

**Table 3.1**

| k | m | a | b | n = km+a+b |
|---|---|---|---|---|
| 5 | 8 | 1 | 1 | 42 |
| 5 | 9 | 8 | 5 | 58 |
| 5 | 9 | 9 | 8 | 62 |
| 8 | 9 | 9 | 1 | 82 |

**Table 3.2**

**Lemma 3.9** There exists a HSOLSSOM($6^{5n}$) for n ≥ 5.

Proof: From Theorem 1.2 and Lemma 3.2, we have a HSOLSSOM($30^n$) for n ≥ 5 and n ∉ {8, 12, 14, 18, 22, 24, 28, 32}. Filling in holes with a HSOLSSOM($6^5$) gives HSOLSSOM($6^{5n}$). Lemmas 3.7 and 3.8 take care of the remaining cases except n = 8 and 12. Applying Construction 2.7 with h = 6, n = 8 and m = 5, we get a HSOLSSOM($30^8$). Filling in holes with a HSOLSSOM($6^5$) gives a HSOLSSOM($6^{40}$). Finally, delete one point from a TD(11, 11). We have an 11-GDD of type $10^{12}$. Giving weight 3 to each point we get a HSOLSSOM($30^{12}$). This takes care of the last case n = 12. □

**Lemma 3.10** There exists a HSOLSSOM($6^n$) for $n \equiv 3 \pmod 4$, $n \geq 5$ and $n \notin \{7, 11, 19, 23, 27, 39\}$.

Proof: From Lemmas 3.7 and 3.9, we need only deal with the cases n = 47, 67; 31, 51, 71; 59, 79; 43, 63 and 83. First, we apply Lemma 3.3 with k = 5 and the other parameters shown in Table 3.3, where the case n = 83 is done by Lemma 3.5 with k = 5.

| d | m | a | b | n = km+a+b+d |
|---|---|---|---|---|
| 0 | 9 | 1 | 1 | 47 |
| 0 | 9 | 5 | 1 | 51 |
| 0 | 9 | 9 | 5 | 59 |
| 0 | 9 | 9 | 9 | 63 |
| 0 | 15 | 8 | 0 | 83 |

**Table 3.3**

From [4] we have a 5-GDD of type $2^{31}$. Giving weight three to each point solves the case n = 31. Add six new points to a HSOLSSOM($42^6$) and fill in holes with a HSOLSSOM($6^8$). This solves the case n = 43. In a similar fashion, HSOLSSOMs of types $42^{10}$ and $78^6$ lead to HSOLSSOMs of types $6^{71}$ and $6^{79}$ respectively.

Apply Theorem 2.2 with q = 7, m = 4 and $e_1 = 1$. We get a HSOLSSOM($4^6 6^1$). Add one new point and fill in the size 4 holes. We have an ISOLSSOM(31, 7). Further apply Theorem 2.3 with q = 13, m = 24 and $e_1 = \ldots = e_6 = 7$. This gives a HSOLSSOM ($24^{13} 84^1$). Adding six new points and filling in holes with a HSOLSSOM($6^5$) and a HSOLSSOM($6^{15}$) solves the case n = 67. □

**Lemma 3.11** There exists a HSOLSSOM($6^n$) for $n \equiv 0 \pmod 4$, $n \geq 5$ and $n \notin \{12, 16, 20, 24, 32\}$.

Proof: From Lemmas 3.7 and 3.9, we need only deal with the cases n = 28, 48, 68, 88; 52, 72, 92; 36, 56, 76, 96; 44, 64 and 84. First, we apply Lemma 3.3 with k = 5 and other parameters shown in Table 3.4. This leaves the cases n = 28, 36, 64 and 84.

| d | m | a | b | n = 5m+a+b+d |
|---|---|---|---|---|
| 1 | 7 | 4 | 4 | 44 |
| 1 | 8 | 7 | 0 | 48 |
| 1 | 8 | 7 | 4 | 52 |
| 1 | 8 | 7 | 8 | 56 |
| 1 | 12 | 7 | 0 | 68 |
| 1 | 12 | 7 | 4 | 72 |
| 1 | 12 | 7 | 8 | 76 |
| 1 | 16 | 7 | 0 | 88 |
| 1 | 16 | 4 | 7 | 92 |
| 1 | 16 | 8 | 7 | 96 |

**Table 3.4**

From [4] we have a BIB design with 169 points and block size 7. Deleting one point gives a 7-GDD of type $6^{28}$. Giving weight one to each point solves the case n = 28. Add six new points to a HSOLSSOM($42^5$) and fill in holes with a HSOLSSOM($6^8$). This solves the case n = 36. Start with a HSOLSSOM($48^8$) and fill in holes with a HSOLSSOM($6^8$). This solves the case n = 64. In a similar way, start with a HSOLSSOM ($84^6$) and fill in holes with a HSOLSSOM($6^{14}$). This solves the case n = 84. □

Combining the results in Theorem 1.2 (2), Lemmas 3.8, 3.10 and 3.11, we have the following theorem.

**Theorem 3.12** There exists a HSOLSSOM($6^n$) for $n \geq 5$ and $n \notin \{6, 7, 10, 11, 12, 16, 18, 19, 20, 22, 23, 24, 27, 32, 38, 39\}$.

We can now update the existence results in Theorems 1.1 and 1.2 as follows.

**Main Theorem** If h is an odd integer, then a HSOLSSOM($h^n$) exists if and only if $n \geq 5$ is odd except possibly when $h = 3$ and $n \in \{19, 23, 27\}$. If h is an even integer, then a HSOLSSOM($h^n$) exists for all $n \geq 5$ except possibly when

(1) $h \equiv 2 \pmod 4$, $h \neq 6$, and $n \in \{8, 12, 14, 18, 22, 24, 28, 32\}$, and

(2) $h = 6$ and $n \in \{6, 7, 10, 11, 12, 16, 18, 19, 20, 22, 23, 24, 27, 32, 38, 39\}$.

**Note added:** Since this paper was submitted for publication, four new HSOLSSOM($6^n$) have been found (see [1]). The possible exceptions n = 10, 11, 16, and 20 for h = 6 in the Main Theorem above have now been removed.

## 4. Three new SOLSSOMs

In this section, we shall use the previous results and techniques to construct three new SOLSSOMs. It is known that a HSOLSSOM($1^n$) is equivalent to an idempotent SOLSSOM(n), which exists if and only if $n \geq 5$ is odd (see [9], [14], [16]). A SOLSSOM is called unipotent if the symmetric orthogonal mate has a constant diagonal. It is known ([9], [14], [7], [2]) that a unipotent SOLSSOM(n) exists if and only if $n \geq 4$ is even, except n = 6 and possibly excepting n = 10, 14, 46, 54, 58, 66, 70.

**Lemma 4.1** There exist unipotent SOLSSOMs of orders n = 46, 54 and 58.

Proof: From Lemma 3.1 there is a HSOLSSOM($3^{15}$). Add a new point to it and fill in holes with a unipotent SOLSSOM(4). This solves the first case.

Delete four points in a group of a TD(6, 5). We have a {5, 6}-GDD of type $5^5 1^1$. Give weight two to each point to get a HSOLSSOM($10^5 2^1$). Fill in holes with an ISOLSSOM(12, 2) (see [8]) and a SOLSSOM(4). This gives the second case. Similar construction works for the third case. In this case, we need a {5, 6}-GDD of type $5^5 3^1$, a HSOLSSOM($10^5 6^1$) and a SOLSSOM(8). □

We can now update the existence results of SOLSSOMs in [2, Theorem 5.1].

**Theorem 4.2** A SOLSSOM(n) exists for all positive integers n, with the exception of n = 2, 3, 6 and the possible exception of n = 10, 14, 66, 70, where the SOLSSOM is idempotent if n is odd and is unipotent if n is even.

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