ON DEFECTIVE COLOURINGS OF COMPLEMENTARY GRAPHS

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ABSTRACT: A graph is **(m,k)-colourable** if its vertices can be coloured with m colours such that the maximum degree of the subgraph induced on vertices receiving the same colour is at most k. The **k-defective chromatic number** $\chi_k(G)$ of a graph G is the least positive integer m for which G is (m,k)-colourable. In this paper we obtain a sharp upper bound for $\chi_1(G) + \chi_1(\overline{G})$ whenever G has no induced subgraph isomorphic to P_4, a path of order four. For general k, we obtain a weak upper bound for $\chi_k(G) + \chi_k(\overline{G})$. Furthermore we will present a sharp lower bound for the product $\chi_k(G).\chi_k(\overline{G})$ in terms of some generalized Ramsey numbers and discuss the associated realizability problem for the 1-defective chromatic number.

1. Introduction

All graphs considered in this paper are undirected, finite, loopless and have no multiple edges. For the most part we follow the notation of Chartrand and Lesniak [3]. For a graph G, we denote the vertex set and

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the edge set of G by V(G) and E(G) respectively. P_n is a path of order n and $\overline{G}$ is the complement of G. For a subset U of V(G), the subgraph of G induced on U is denoted by G[U].

The generalized Ramsey number R(K(1,m),K(1,n)) is the smallest positive integer p such that for every graph G of order p, either G contains a vertex whose degree is at least m or $\overline{G}$ contains a vertex with degree at least n.

A graph is said to be **P_4-free**, if it does not contain P_4 as an induced subgraph. A subset U of V(G) is said to be **k-independent** if the maximum degree of G[U] is at most k. A graph is **(m,k)-colourable** if its vertices can be coloured with m colours such that the subgraph induced on vertices receiving the same colour is k-independent. Sometimes we refer to an (m,k)-colouring of G as a **k-defective colouring** of G. Note that any (m,k)-colouring of a graph G partitions the vertex set of G into m subsets V_1, V_2, ..., V_m such that every V_i is k-independent. The **k-defective chromatic number** $\chi_k(G)$ of G is the least positive integer m for which G is (m,k)-colourable. Note that $\chi_0(G)$ is the usual chromatic number of G. Clearly $\chi_k(G) \leq \left\lceil \frac{p}{k+1} \right\rceil$, where p is the order of G. If $\chi_k(G) = m$ then G is said to be **(m,k)-chromatic**. In addition, if $\chi_k(G-v) = m-1$ for every vertex v of G then G is said to be **(m,k)-critical.**

These concepts have been studied by several authors. Hopkins and Staton [6] refer to a k-independent set as a k-small set. Maddox [8,9] and Andrews and Jacobson [2] refer to the same as a k-dependent set. The k-defective chromatic number has been investigated by Frick [4]; Frick and Henning [5]; Maddox [8,9]; Hopkins and Staton [6] under the name k-partition number; Andrews and Jacobson [2] under the name k-chromatic number.

The k-defective chromatic number is a generalization of the chromatic number of a graph which is related to the point partition number $\rho_k(G)$ defined by Lick and White [7]. It is well known that $\chi_k(G) \geq \rho_k(G)$. Lick and White [7] established that

$$\rho_k(G) + \rho_k(\overline{G}) \leq \frac{p-1}{k+1} + 2,$$

for a graph G of order p. A natural question that arises is whether the above upper bound is approximately the right bound for $\chi_k(G) + \chi_k(\overline{G})$. We investigate this problem in this paper.

The Nordhaus-Gaddum [10] problem associated with the parameter χ_k is to find sharp bounds for $\chi_k(G) + \chi_k(\overline{G})$ and $\chi_k(G).\chi_k(\overline{G})$ where G is a graph of order p. Maddox [8] investigated this problem and has shown that if either G or $\overline{G}$ is triangle free, then $\chi_k(G) + \chi_k(\overline{G}) \leq 5\left\lceil \frac{p}{3k+4} \right\rceil$, where p is the order of G. When k = 1 he improved the above bound to $6\left\lceil \frac{p}{9} \right\rceil$. Maddox [8] suggested the following conjecture:

For a graph G of order p,

$$\chi_k(G) + \chi_k(\overline{G}) \leq \left\lceil \frac{p-1}{k+1} \right\rceil + 2 .$$

Achuthan et al.[1] have proved that for any graph G of order p,

$$\chi_1(G) + \chi_1(\overline{G}) \leq \frac{2p+4}{3} .$$

In this paper we will investigate the Nordhaus-Gaddum problem for the k-defective chromatic number of a graph. In Section 2 we will prove that $\chi_1(G) + \chi_1(\overline{G}) \leq \left\lfloor \frac{p}{2} \right\rfloor + 2$ for a P_4-free graph G of order p.

Note that this verifies the above conjecture of Maddox for the case k = 1 over the subclass of P_4-free graphs of order p. We will also establish a weak upper bound for $\chi_k(G) + \chi_k(\overline{G})$, for $k \geq 1$ where G is a graph of order p. In Section 3 we disprove the conjecture of Maddox [8] and in Section 4 we will present a sharp lower bound for $\chi_k(G).\chi_k(\overline{G})$. In the final section we will study the following realizability problem for the 1-defective chromatic number :

Given a positive integer p, determine integer pairs x and y such that there exists a P_4 -free graph G of order p with $\chi_1(G) = x$ and $\chi_1(\overline{G}) = y$.

In all the figures of this paper, we follow the convention that a solid line between two sets X and Y of vertices represents the existence of all possible edges between X and Y.

2. Upper bound for the sum

We need the following theorem to prove our results.

Theorem 1(Seinsche [11]):

Let G be a graph. The following statements are equivalent.

1. G has no induced subgraph isomorphic to P_4 .
2. For every subset U of V(G) with more than one element, either G[U] or $\overline{G}$[U] is disconnected. □

Our first result deals with critical graphs.

Lemma 1: Let G be (m,k)-critical. For all $v \in V(G)$ the k-defective chromatic number of every component of G - v is equal to m - 1.

Proof: Let $v \in V(G)$. Since G is (m,k)-critical, $\chi_k(G-v) = m - 1$. If v is not a cut vertex there is nothing to prove. Now let v be a cut vertex of G and let $H_1, H_2, \ldots, H_t$, be the components of G - v. If $\chi_k(H_1) = \chi_k(H_2) = \ldots = \chi_k(H_t)$ then the lemma follows from the criticality of G. Otherwise, let l be an integer, $1 \le l \le t$, such that $\chi_k(H_l) \le m - 2$. From the criticality of G it follows that $G - H_l$ is (m-1,k)-colourable. Consider any (m-1,k)-colouring of the vertices of $G - H_l$ using colours 1,2,...,m - 1. Without loss of generality we can assume that m - 1 is the colour received by the vertex v. Now consider an (m-2,k)-colouring of the graph H_l using the colours 1,2,..., m-2. Note that this is possible since $\chi_k(H_l) \le m - 2$. This produces an (m-1,k)-colouring of G, which contradicts the hypothesis and proves the lemma. □

Lemma 2 : Let G be a graph of order p with vertex disjoint stars S_1, $S_2, \ldots, S_\alpha$ of order k + 2 each. Then

$$\chi_k(\overline{G}) \le \left\lceil \frac{p-\alpha}{k+1} \right\rceil$$

Proof: Clearly $V(S_i)$ is a k-independent set in $\overline{G}$, for each i . Now consider the following colouring of $\overline{G}$: The vertices of S_i are assigned colour i, $1 \le i \le \alpha$; and the remaining $p - (k+2)\alpha$ vertices are coloured using $\left\lceil \frac{p-(k+2)\alpha}{k+1} \right\rceil$ new colours. This is a k-defective colouring of $\overline{G}$ which uses $\left\lceil \frac{p-\alpha}{k+1} \right\rceil$ colours. Thus $\chi_k(\overline{G}) \le \left\lceil \frac{p-\alpha}{k+1} \right\rceil$ □

Theorem 2 : Let G be a P_4 -free graph of order $p \geq 3$. Then

$$\chi_1(G) + \chi_1(\overline{G}) \leq \left\lfloor \frac{p}{2} \right\rfloor + 2.$$

Moreover this bound is sharp.

Proof: We prove the theorem by induction on p. It is clearly true for p = 3 and 4 and hence let $p \geq 5$. Assume that the theorem holds for P_4 -free graphs of order < p. We first observe that for every pair of vertices x and y of G,

$$\chi_1(G - x - y) = \chi_1(G) \text{ or } \chi_1(G) - 1,$$

and

$$\chi_1(\overline{G} - x - y) = \chi_1(\overline{G}) \text{ or } \chi_1(\overline{G}) - 1.$$

Suppose there are vertices x and y such that

$$\chi_1(G - x - y) = \chi_1(G)$$

or

$$\chi_1(\overline{G} - x - y) = \chi_1(\overline{G}).$$

In this case

$$\chi_1(G) + \chi_1(\overline{G}) \leq \chi_1(G - x - y) + \chi_1(\overline{G} - x - y) + 1.$$

Using the induction hypothesis we have

$$\chi_1(G) + \chi_1(\overline{G}) \leq \left\lfloor \frac{p-2}{2} \right\rfloor + 2 + 1 = \left\lfloor \frac{p}{2} \right\rfloor + 2.$$

Henceforth we assume that for all x and $y \in V(G)$,

$$\chi_1(G) = \chi_1(G - x - y) + 1 \quad (1)$$

and

$$\chi_1(\overline{G}) = \chi_1(\overline{G} - x - y) + 1 \quad (2)$$

Since G is P_4 -free, it follows from Theorem 1 that either G or $\overline{G}$ is disconnected. Let us assume without loss of generality that G is disconnected. Let G_1 be a component of G with the largest value of χ_1 and G_2 be the union of all other components of G. Note that $\chi_1(G_1) = \chi_1(G)$. Clearly G_2 has exactly one vertex, for otherwise, we have a contradiction to (1) . Hence let $V(G_2) = \{ w \}$. Since G_1 is connected and P_4 -free, it follows from Theorem 1 that $\overline{G_1}$ is disconnected. Let F_1 , F_2 , ... , F_t be the components of $\overline{G_1}$ such that $\chi_1(F_1) \geq \chi_1(F_2) \geq \ldots \geq \chi_1(F_t)$. If $\chi_1(F_1) = 1$ then $\chi_1(\overline{G}) \leq 2$. In this case

$$\chi_1(G) + \chi_1(\overline{G}) \leq \chi_1(G_1) + 2 \leq \left\lceil \frac{p-1}{2} \right\rceil + 2 = \left\lfloor \frac{p}{2} \right\rfloor + 2.$$

Henceforth we will assume that $\chi_1(F_1) \geq 2$. Let $U \equiv F_2 \cup F_3 \cup \ldots \cup F_t$, $|V(F_1)| = a$, and $|V(U)| = b$. Note that $\chi_1(U) = \chi_1(F_2)$. The graph $\overline{G}$ is depicted in Figure 1 .

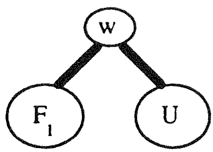

Figure 1: $\overline{G}$

We now consider two cases depending on the value of $\chi_1(U)$.

Case 1: $\chi_1(U) \geq 3$.

Since U is P_4 -free, by the induction hypothesis we have

$$\chi_1(\overline{U}) \leq \left\lfloor \frac{b}{2} \right\rfloor + 2 - \chi_1(U)$$

$$\leq \left\lfloor \frac{b}{2} \right\rfloor - 1.$$

Also note that

$$\chi_1(G) \leq \chi_1(\overline{F_1}) + \chi_1(\overline{U})$$

and

$$\chi_1(\overline{G}) \leq \chi_1(F_1) + 1 .$$

Therefore

$$\chi_1(G) + \chi_1(\overline{G}) \leq \chi_1(F_1) + \chi_1(\overline{F_1}) + \chi_1(\overline{U}) + 1$$

$$\leq \left\lfloor \frac{a}{2} \right\rfloor + 3 + \left\lfloor \frac{b}{2} \right\rfloor - 1$$

$$\leq \left\lfloor \frac{a+b}{2} \right\rfloor + 2 = \left\lfloor \frac{p-1}{2} \right\rfloor + 2.$$

Case 2: $\chi_1(U) \leq 2$

Observe that $\chi_1(F_1 + w) \geq \chi_1(U)$. Firstly if equality occurs in this inequality, then $\chi_1(F_1 + w) = \chi_1(F_1) = \chi_1(U) = 2$, since $\chi_1(F_1) \geq 2$. Consequently there are two vertex disjoint paths Q_1 and Q_2 of length two in F_1 and F_2 respectively. Applying Lemma 2 to the graph $\overline{G_1}$ (of order p - 1) we have $\chi_1(G_1) \leq \left\lceil \frac{p-3}{2} \right\rceil$. Now $\chi_1(G) = \chi_1(G_1) \leq \left\lceil \frac{p-3}{2} \right\rceil$. Since $\chi_1(\overline{G}) \leq \chi_1(F_1) + 1 = 3$, we have

$$\chi_1(G) + \chi_1(\overline{G}) \leq \left\lceil \frac{p-3}{2} \right\rceil + 3 = \left\lfloor \frac{p}{2} \right\rfloor + 2.$$

Henceforth we will assume that $\chi_1(F_1 + w) > \chi_1(U)$.

We will now prove that $\chi_1(\overline{G}) = \chi_1(F_1 + w)$. Firstly observe that $\chi_1(\overline{G}) \geq \chi_1(F_1 + w)$, since $F_1 + w$ is a subgraph of $\overline{G}$. Consider a 1-defective colouring of $F_1 + w$ using $\chi_1(F_1 + w)$ colours. Since $\chi_1(U) < \chi_1(F_1 + w)$ it is possible to colour all the vertices of U with the colours used in the above mentioned 1-defective colouring of $F_1 + w$ except the one given to the vertex w. This provides a 1-defective colouring of $\overline{G}$ with $\chi_1(F_1 + w)$ colours. Thus $\chi_1(\overline{G}) = \chi_1(F_1 + w)$. Now $|V(U)| = 1$, for otherwise, we have a contradiction to (2). Let $V(U) = \{ z \}$.

Since F_1 is connected and P_4 -free, it follows that $\overline{F_1}$ is disconnected. Let $H_1, H_2, \ldots, H_\lambda$ be the components of $\overline{F_1}$. Define $Y \cong H_2 \cup H_3 \cup \ldots \cup H_\lambda$ and let $|V(H_1)| = c$ and $|V(Y)| = d$. Note that $c + d = p - 2$.

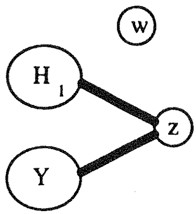

Figure 2: G

We observe that G - w is critical, for otherwise, if $\chi_1(G - w - u) = \chi_1(G - w)$ for some vertex u then we have a contradiction to (1) since $\chi_1(G - w) = \chi_1(G)$. Now from Lemma 1 we have,

$$\chi_1(H_1) = \chi_1(H_2) = \ldots = \chi_1(H_\lambda) = \chi_1(G\text{-}w) - 1 = \chi_1(G) - 1.$$

Also since $\chi_1(\overline{G}) \leq \chi_1(\overline{H_1}) + \chi_1(\overline{Y}) + 1$, we have

$$\chi_1(G) + \chi_1(\overline{G}) \leq \chi_1(H_1) + \chi_1(\overline{H_1}) + \chi_1(\overline{Y}) + 2$$

$$\leq \left\lfloor \frac{c}{2} \right\rfloor + \chi_1(\overline{Y}) + 4. \qquad (3)$$

Firstly let $\chi_1(Y) \geq 3$. Since Y is P_4 -free we have

$$\chi_1(\overline{Y}) \leq \left\lfloor \frac{d}{2} \right\rfloor - 1.$$

Incorporating this inequality in (3) we have

$$\chi_1(G) + \chi_1(\overline{G}) \leq \left\lfloor \frac{c}{2} \right\rfloor + \left\lfloor \frac{d}{2} \right\rfloor + 3$$

$$\leq \left\lfloor \frac{c+d}{2} \right\rfloor + 3 = \left\lfloor \frac{p}{2} \right\rfloor + 2.$$

This proves the theorem in the case $\chi_1(Y) \geq 3$. Henceforth let us assume that $\chi_1(Y) \leq 2$. Note that $\chi_1(Y) = \chi_1(H_1) = \chi_1(H_2) = \dots = \chi_1(H_\lambda)$.

If $\chi_1(Y) = 1$ then clearly $\chi_1(G) \leq 2$. Let $u \in V(H_1)$ and $v \in V(H_2)$. Then $G[\{u,v,z\}]$ contains a path of length 2. Again by Lemma 2, $\chi_1(\overline{G}) \leq \left\lceil \frac{p-1}{2} \right\rceil = \left\lfloor \frac{p}{2} \right\rfloor$. Thus $\chi_1(G) + \chi_1(\overline{G}) \leq \left\lfloor \frac{p}{2} \right\rfloor + 2$ in this case.

Finally let $\chi_1(Y) = 2$. Clearly $\chi_1(G_1) \leq 3$. Since $\chi_1(H_i) = 2$ for each i, H_i contains a path Q_i of length 2. Note that $V(Q_1)$ and $V(Q_2) \cup \{z\}$ are 1-independent in $\overline{G}$. Now assign colour 1 to the vertices of $V(Q_1)$, colour 2 to the vertices of $V(Q_2) \cup \{z\}$ and $\left\lceil \frac{p-7}{2} \right\rceil$ new colours to the remaining p - 7 vertices of $\overline{G}$. This is a 1-defective colouring of $\overline{G}$ which uses $\left\lceil \frac{p-3}{2} \right\rceil$ colours. Thus $\chi_1(\overline{G}) \leq \left\lceil \frac{p-3}{2} \right\rceil$. Combining this with the inequality $\chi_1(G) \leq 3$ we have the required upper bound.

To prove the sharpness let $G \cong K(1,p-1)$. Clearly $\chi_1(G) = 2$ and $\chi_1(\overline{G}) = \left\lfloor \frac{p}{2} \right\rfloor$. This completes the proof of the theorem. □

Recall the following conjecture of Maddox [8] concerning the 1-defective chromatic number :

For a graph G of order p,

$$\chi_1(G) + \chi_1(\overline{G}) \leq \left\lceil \frac{p-1}{2} \right\rceil + 2.$$

Theorem 2 verifies this conjecture for the subclass of P_4-free graphs of order p.

Next we establish a weak upper bound for $\chi_k(G) + \chi_k(\overline{G})$ for all $k \geq 1$.

Theorem 3: Let G be a graph of order p. Then

$$\chi_k(G) + \chi_k(\overline{G}) \leq \frac{2p+2k+4}{k+2}.$$

Proof : Consider a partition of V(G) into k-independent sets $V_1, V_2, \ldots$ constructed as follows:

- V_1 is the largest k-independent set of G.
- Having defined the i^{th} k-independent set V_i, the $(i+1)^{th}$ set V_{i+1} is defined as the largest k-independent set in the subgraph induced on $V(G) - \bigcup_{l=1}^{i} V_l$.
- Repeat the above process until we can not proceed any further.

Clearly this procedure produces a partition of V(G) into, say m, k-independent sets $V_1, V_2, \ldots, V_m$ with the following properties:

(i) $|V_1| \geq |V_2| \geq \ldots \geq |V_m|$

and

(ii) $|V_{m-1}| \geq k+1$.

Observe that $\chi_k(G) \leq m$ and we will now prove that

$$\chi_k(\overline{G}) \leq \frac{p+k+2-m}{k+1}. \qquad (4)$$

Let $x_i \in V_i$ for $i \geq 2$. Note that $G[V_{i-1} \cup \{ x_i \}]$ contains a star $S_i \cong K(1,k+1)$, for otherwise, $V_{i-1} \cup \{x_i\}$ is a k-independent set, contradicting the maximality of $|V_{i-1}|$. Now we define r to be the smallest positive integer i such that $|V_i| \leq k + 1$. If no such r exists then let $r = m$. Note that $|V_{r-1}| \geq k + 2$. We consider two cases.

Case 1: $r = m$

Since $|V_i| \geq k + 2$ for $2 \leq i \leq m - 1$, the stars $S_i \cong K(1,k+1)$, $i = 2,3,\ldots,m$ of G can be chosen to be vertex disjoint. Using Lemma 2 we have

$$\chi_k(\overline{G}) \leq \left\lceil \frac{p-(m-1)}{k+1} \right\rceil \leq \frac{p-m+k+2}{k+1} .$$

This establishes (4) in this case.

Case 2 : $r \leq m - 1$

Note that in this case $|V_i| = k + 1$ for $r \leq i \leq m - 1$. Since $|V_m| \geq 1$ we have $|\bigcup_{i=r}^{m} V_i| \geq (m - r)(k + 1) + 1$.

Clearly $\bigcup_{i=r}^{m} V_i$ is k-independent in $\overline{G}$, for otherwise, $\overline{G}[\bigcup_{i=r}^{m} V_i]$ has a star $S \cong K(1,k + 1)$ and thus $V(S)$ forms a k-independent set of cardinality $k + 2$ in G, contradicting the maximality of $|V_r|$. Again as in Case 1, since $|V_i| \geq k + 2$ for $i = 1,2,\ldots,r-1$, the stars $S_2, S_3, \ldots, S_r$ can be chosen to be vertex disjoint. Now we provide a k-defective colouring of $\overline{G}$ as follows:

- colour the vertices of S_i with colour i, $2 \leq i \leq r$.

- colour the vertices of $\bigcup_{i=r}^{m} V_i - S_r$ with colour 1. Note that $|\bigcup_{i=r}^{m} V_i - S_r| \geq (m - r)(k + 1)$.

- colour the remaining α vertices of $\overline{G}$ arbitrarily, using $\left\lceil \frac{\alpha}{k+1} \right\rceil$ new colours where $\alpha = p - (r - 1)(k + 2) - |\bigcup_{i=r}^{m} V_i - S_r|$.

Note that $\alpha \leq p - (r - 1)(k + 2) - (m - r)(k + 1)$.

Thus

$$\chi_k(\overline{G}) \leq \left\lceil \frac{p-(r-1)(k+2)-(m-r)(k+1)}{k+1} \right\rceil + r$$

$$\leq \frac{p+k+2-m}{k+1}.$$

This proves (4).

Now from (4) and the inequality $\chi_k(G) \leq m$, we have

$$(k + 1)\chi_k(\overline{G}) + \chi_k(G) \leq p + k + 2.$$

Now reversing the roles of G and $\overline{G}$, we get

$$\chi_k(\overline{G}) + (k + 1)\chi_k(G) \leq p + k + 2.$$

Combining these two inequalities we have the required inequality. □

3. Counter example to the conjecture of Maddox

In this section we will construct a graph G of order p such that $\chi_k(G) + \chi_k(\overline{G}) = \left\lceil \frac{p-1}{k+1} \right\rceil + 3$, thus disproving the conjecture of Maddox[8] which states that for a graph G of order p,

$$\chi_k(G) + \chi_k(\overline{G}) \leq \left\lceil \frac{p-1}{k+1} \right\rceil + 2.$$

Lemma 3 : Suppose $k \geq 2$ and $m \geq 0$ are integers. Let G be a graph of order $(m + 3)(k + 1)$ shown in Figure 3, where $G[A_1] \cong \overline{K}_k$, $G[A_2] \cong$

$G[A_3] \cong K_k$, $G[A_4] \cong \overline{K}_2$ and $G[A_5] \cong K_{m(k+1)+1}$. Then $\chi_k(G) = m + 3$.

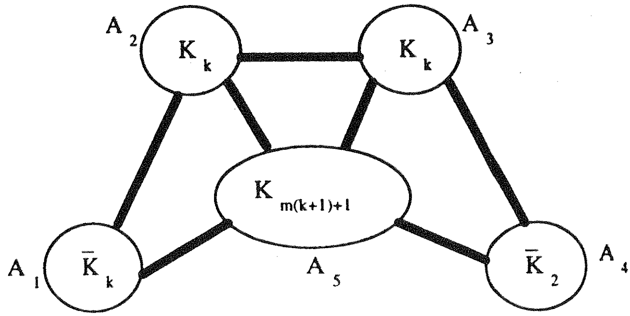

Figure 3: G

Proof : Firstly $\chi_k(G) \le m + 3$, since G has $(m + 3)(k + 1)$ vertices. If possible let $\chi_k(G) \le m + 2$ and consider a partition of V(G) into $m + 2$ k-independent sets $V_1, V_2, \ldots, V_{m+2}$ such that V_1 is a largest set. Since $|V_1| \ge k + 2$ and the elements of A_5 are adjacent to every other vertex of G, it follows that $A_5 \cap V_1 = \emptyset$, $A_5 \cap V_i \ne \emptyset$ for $i \ge 2$ and $|V_i| \le k+1$ for $i \ge 2$. Thus $|V_1| \ge 2k + 2$ and $V_1 \subseteq A_1 \cup A_2 \cup A_3 \cup A_4$. Now if $A_2 \cap V_1 = \emptyset$, then $V_1 = A_1 \cup A_3 \cup A_4$, which is not k-independent, and therefore a contradiction. On the other hand, if $A_2 \cap V_1 \ne \emptyset$ then $|V_1 \cap (A_1 \cup A_2 \cup A_3)| \le k + 1$, so that $|V_1| \le k + 3$. Thus we have $2k + 2 \le |V_1| \le k + 3$ which implies $k \le 1$, a contradiction to our assumtion that $k \ge 2$. This completes the proof of the lemma. □

Lemma 4: Suppose $k \ge 1$ and $t \ge 0$ are integers. Let G be a graph of order $(t + 3)(k + 1)$ shown in Figure 4, where $G[A_1] \cong G[A_4] \cong \overline{K}_k$, $G[A_2] \cong K_k$, $G[A_3] \cong K_2$ and $G[A_5] \cong K_{t(k+1)+1}$. Then $\chi_k(G) = t + 3$.

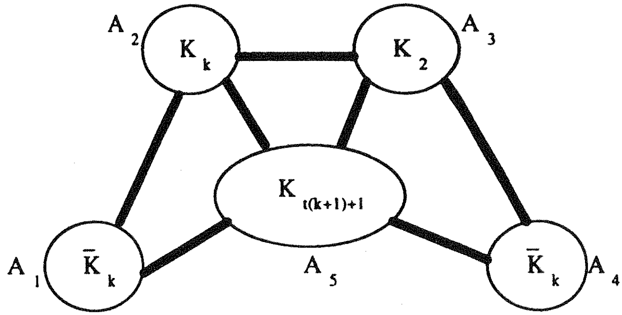

Figure 4: G

Proof: The proof of Lemma 4 is identical to that of Lemma 3, except that $A_2 \cap V_1 = \emptyset$ impies $|V_1 \cap (A_1 \cup A_3)| \le k + 1$ which in turn implies that $|V_1| \le 2k + 1$, contradicting the inequality $|V_1| \ge 2k + 2$. □

Lemma 5 : Let $G \cong K_{2m+1} + C_5$. Then $\chi_1(G) = m + 3$.

Proof: Since the order of G is 2m + 6, it follows that $\chi_1(G) \le m + 3$. If possible let $\chi_1(G) \le m + 2$ and consider a partition of V(G) into 1-independent sets V_1, V_2, ..., V_{m+2} . Without loss of generality assume that $|V_1| \ge |V_2| \ge \ldots \ge |V_{m+2}|$. Since $\overline{G} \cong C_5 \cup \overline{K}_{2m+1}$, any 1-independent set of G has cardinality at most 3. Therefore $|V_1| \le 3$. Again if $|V_2| = 3$ then $\overline{G}$ would have two vertex disjoint paths of length 2 each, which is impossible. Therefore $|V_2| \le 2$. Thus

$$2m + 6 = |V(G)| = \sum_{i=1}^{m+2} |V_i| \le 2m + 5,$$

which is absurd. This proves $\chi_1(G) \ge m + 3$, completing the proof of the lemma. □

We will now present a graph which disproves the conjecture of Maddox [8].

Theorem 4 : Let $k \geq 2$, $t \geq 0$ and $m \geq 0$ be integers and G a graph of order $(t + m + 3)(k + 1) + 1$ shown in Figure 5, where $G[A_1] \cong \overline{K}_k$, $G[A_2] \cong G[A_3] \cong K_k$, $G[A_4] \cong \overline{K}_2$, $G[A_5] \cong K_{m(k+1)+1}$ and $G[A_6] \cong \overline{K}_{t(k+1)+1}$. Then

$$\chi_k(G) + \chi_k(\overline{G}) = m + t + 6.$$

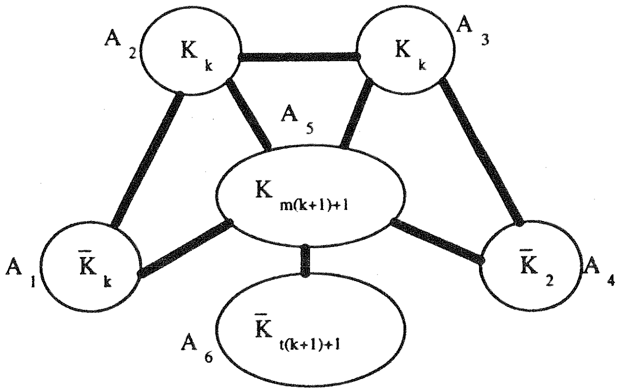

Figure 5: G

Proof : It is easy to see that $\chi_k(G) \leq m + 3$, since the vertices of $A_2 \cup A_3 \cup A_5$ can be arbitrarily coloured with $m + 2$ colours and all the vertices of $A_1 \cup A_4 \cup A_6$ can be coloured with a new colour. Since G contains the graph of Lemma 3 as a subgraph it follows that $\chi_k(G) \geq m + 3$. Thus $\chi_k(G) = m + 3$.

Note that $\overline{G}$ is the disjoint union of the graph of Lemma 4 and a $\overline{K}_{m(k+1)+1}$. Thus from Lemma 4, we have

$\chi_k(\overline{G}) = t + 3$. Hence $\chi_k(G) + \chi_k(\overline{G}) = m + t + 6$. □

Theorem 5 : Let G be the graph of Figure 6 where $G[X] \cong \overline{K}_{2t+1}$, $G[Y] \cong K_{2m+1}$ and $G[Z] \cong C_5$. Then

$$\chi_1(G) + \chi_1(\overline{G}) = m + t + 6.$$

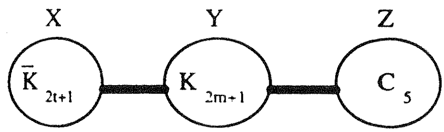

Figure 6: G

Proof: Firstly colour the vertices of Y using m + 1 colours. Now the vertices of $X \cup Z$ can be coloured with two new colours. This is possible since there are no edges between X and Z and $\chi_1(C_5) = 2$. Thus $\chi_1(G) \leq m + 3$. Also $\chi_1(G) \geq \chi_1(G[Y \cup Z]) = m + 3$ (by Lemma 5). Hence $\chi_1(G) = m + 3$.

Similarly using Lemma 5 one can show that $\chi_1(\overline{G}) = t + 3$. This proves that $\chi_1(G) + \chi_1(\overline{G}) = m + t + 6$. □

Recall the conjecture of Maddox [8]:

For a graph G of order p,

$$\chi_k(G) + \chi_k(\overline{G}) \leq \left\lceil \frac{p-1}{k+1} \right\rceil + 2.$$

Simple counting shows that the graphs of Theorems 4 and 5 form counter-examples to the conjecture for $k \geq 2$ and $k = 1$, respectively. It is also easy to see that these graphs have P_4 as an induced subgraph. A natural question that arises is : Does there exist a P_4-free graph G of order p such that $\chi_k(G) + \chi_k(\overline{G}) \geq \left\lceil \frac{p-1}{k+1} \right\rceil + 3$ for $k \geq 2$?

4. Lower bound for the product

In this section we will provide a sharp lower bound for the product $\chi_k(G).\chi_k(\overline{G})$ in terms of the generalized Ramsey number $R(K(1,k+1),K(1,k+1))$.

Theorem 6 (Chartrand and Lesniak[3], p. 315)
Let k be a positive integer. Then

$$R(K(1,k+1),K(1,k+1)) = \begin{cases} 2k+1, \text{ if } k \text{ is odd} \\ 2k+2, \text{ otherwise.} \end{cases} \qquad \square$$

For notational convenience we denote $R(K(1,k+1),K(1,k+1))$ by R. From the definition of the generalized Ramsey number R it follows that for any positive integer $t \leq R-1$, there exists a graph H of order t such that neither H nor $\overline{H}$ contains a vertex of degree $k+1$. We refer to such a graph as a Ramsey graph and denote it by H[t].

Lemma 6: Let G be a graph of order p. If $\chi_k(G) = 1$, then

$$\chi_k(\overline{G}) \geq \frac{p}{R-1}.$$

Proof : Let $\chi_k(\overline{G}) = m$ and consider an (m,k)-colouring of $\overline{G}$. Let $V_1, V_2, \ldots, V_m$ be a partition of $V(\overline{G})$ into k-independent sets such that $|V_1| = \max_i |V_i|$. Note that $|V_1| \geq \frac{p}{m}$. Since V_1 is k-independent in both G and $\overline{G}$, it follows from the definition of R that $|V_1| \leq R-1$.
Thus $\chi_k(\overline{G}) = m \geq \dfrac{p}{R-1}$. $\square$

Theorem 7 : Let G be a graph of order p. Then

$$\chi_k(G).\chi_k(\overline{G}) \geq \left\lceil \frac{p}{R-1} \right\rceil$$

Moreover this bound is sharp.

Proof : Let $\chi_k(G) = m$ and $V_1, V_2, \ldots, V_m$ be a partition of V(G) into k-independent sets such that $|V_1| = \max_i |V_i|$.

Since V_1 is k-independent in G we have $\chi_k(G[V_1]) = 1$. Thus using Lemma 6,

$$\chi_k(\overline{G}) \geq \chi_k(\overline{G}[V_1]) \geq \frac{|V_1|}{R-1}.$$

Combining the above inequality with the fact that $|V_1| \geq \frac{p}{m}$ we have

$$\chi_k(G).\chi_k(\overline{G}) \geq \left\lceil \frac{p}{R-1} \right\rceil.$$

We will now establish the sharpness of the above inequality. For notational convenience let us write $\left\lceil \frac{p}{R-1} \right\rceil = \lambda$. Define G to be the disjoint union of λ Ramsey graphs $H_1, H_2, \ldots, H_\lambda$ where

$$|V(H_i)| = \begin{cases} R-1, \text{ for } i, \ 1 \leq i \leq \lambda-1, \\ R-1, \text{ if } i=\lambda \text{ and } R-1 \text{ divides } p, \\ p-\left\lfloor \frac{p}{R-1} \right\rfloor (R-1), \text{ otherwise.} \end{cases}$$

It is easy to see that the order of G is p and $\chi_k(G) = 1$. From Lemma 6, $\chi_k(\overline{G}) \geq \lambda$. To prove the reverse inequality, assign colour i to the vertices of H_i for $i = 1,2,\ldots,\lambda$. Since $V(H_i)$ is k-independent in $\overline{G}$, this provides a (λ,k)-colouring of $\overline{G}$. Thus $\chi_k(\overline{G}) = \lambda$. This completes the proof of the theorem. □

Remark 1: In particular we have, $\chi_1(G).\chi_1(\overline{G}) \geq \frac{p}{2}$, since

$$R(K(1,2),K(1,2)) = 3. \qquad \square$$

5. Realizability problem

In this section we will address the realizability problem associated with the parameter χ_1 over the class of P_4-free graphs.

Problem : Given integers x, y and $p \geq 3$, determine necessary and sufficient conditions for the existence of a P_4-free graph G of order p such that $\chi_1(G) = x$ and $\chi_1(\overline{G}) = y$.

Let x and y be integers such that $x \leq \left\lceil \frac{p}{2} \right\rceil$ and $y \leq \left\lceil \frac{p}{2} \right\rceil$. Consider the following inequalities:

$$x + y \leq 2 + \frac{p}{2} \tag{5}$$

$$xy \geq \frac{p}{2} \tag{6}$$

From Theorem 2 and Remark 1, it follows that (5) and (6) are necessary for the existence of a P_4-free graph G of order p with $\chi_1(G) = x$ and $\chi_1(\overline{G}) = y$. In this section we will establish the sufficiency.

Theorem 8 : Let $x \leq \left\lceil \frac{p}{2} \right\rceil$, $y \leq \left\lceil \frac{p}{2} \right\rceil$ and $p \geq 3$ be integers such that (5) and (6) hold. Then there is a P_4-free graph G of order p with $\chi_1(G) = x$ and $\chi_1(\overline{G}) = y$.

Proof : Without loss of generality let $x \leq y$. From (5) we have $p \geq 2x + 2y - 4$.

Case 1 : $p = 2x + 2y - 3$ or $2x + 2y - 4$.

Firstly if $x = 1$, then $y = \frac{p+1}{2}$. In this case the graph $\overline{K}_p$ is the required graph.

Next let $x \geq 2$. Consider the graph $G \equiv (K_{2x-3} + \overline{P}_3) \cup \overline{K}_{2y-4+\delta}$, where $\delta = 0$ or 1 according as p is even or odd. It is easy to verify that G is a P_4 -free graph, $\chi_1(G) = x$ and $\chi_1(\overline{G}) = y$.

Case 2 : $2(x + y - 1) \leq p \leq 2xy$

Let $\alpha_1, \alpha_2, \ldots, \alpha_y$ be integers satisfying the following conditions:

$$\alpha_1 = 2x,$$

$$2 \leq \alpha_i \leq 2x, \quad 2 \leq i \leq y,$$

and

$$\sum_{i=1}^{y} \alpha_i = p .$$

It is easy to check that such integers $\alpha_1, \alpha_2, \ldots, \alpha_y$ always exist. For example, the numbers defined below satisfy the required conditions.

$$\alpha_1 = 2x,$$

$$\alpha_i = t + 3, \quad 2 \leq i \leq s + 1,$$

and

$$\alpha_i = t + 2, \quad s + 2 \leq i \leq y,$$

where $p - 2(x + y - 1) = t(y - 1) + s$, $0 \leq s < y - 1$.

Now let $G \equiv K_{\alpha_1} \cup K_{\alpha_2} \cup \ldots \cup K_{\alpha_y}$. Note that G is P_4 -free. Clearly $\chi_1(G) = \chi_1(K_{\alpha_1}) = x$. Since G contains a 1-independent set of cardinality $2y$, from Lemma 6, we have $\chi_1(\overline{G}) \geq y$. Also it is easy to check that $\overline{G}$ is $(y,1)$-colourable. Thus $\chi_1(\overline{G}) = y$. This completes the proof of the theorem. □

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