A NOTE ON k-EXTENDABLE GRAPHS AND INDEPENDENCE NUMBER

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ABSTRACT:

An even order graph G with a perfect matching is **k-extendable** if for every matching M of size k in G, there exists a perfect matching in G containing all the edges of M. In this note, we establish a necessary and sufficient condition for a graph to be k-extendable in terms of its independence number.

All graphs considered in this note are finite, connected, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [1]. Thus, G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices, minimum degree $\delta(G)$ and independence number $\alpha(G)$. For $V' \subseteq V(G)$, $G[V']$ denotes the subgraph induced by V'. The join $G \vee H$ of disjoint graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. The number of odd components of a graph G is denoted by $o(G)$.

Let G be a simple connected graph on 2n vertices with a perfect matching. For a positive integer k, $1 \le k \le n - 1$, G is **k-extendable** if for every matching M of size k in G, there exists a perfect matching in G containing all the edges of M. Observe that $K_{n,n}$ and K_{2n} are k-extendable for all k, $1 \le k \le n - 1$. The cycle C_{2n} of order 2n is

1-extendable but not 2-extendable. k-extendable graphs have been studied by many authors. An excellent survey is the paper of Plummer [2].

In this note we prove the following theorem:

Theorem 1: Let G be a graph on 2n vertices with $\delta(G) \geq n + k - 1$ and k any positive integer such that $\frac{n}{2} \leq k \leq n - 2$ and $n - k$ is even. Then G is k-extendable if and only if $\alpha(G) \leq n - k$.

Our proof makes use of the following two well-known results.

Theorem 2: Tutte's Theorem (see Bondy and Murty [1] p.76)

A graph G has a perfect matching if and only if

$$o(G - S) \leq |S|, \qquad \text{for all } S \subset V(G). \quad \square$$

Theorem 3: Dirac's Theorem (see Bondy and Murty [1] p.54)

If G is a simple graph with $\nu(G) \geq 3$ and $\delta(G) \geq \frac{1}{2}\nu(G)$, then G is hamiltonian. $\square$

Our first result provides an upper bound on $\alpha(G)$ for a k-extendable graph G on 2n vertices whose minimum degree is at least $2k + 1$ for $\frac{n}{2} \leq k \leq n - 1$.

Lemma 4: If G is a k-extendable graph on 2n vertices with $\frac{n}{2} \leq k \leq n - 1$ and $\delta(G) \geq 2k + 1$, then $\alpha(G) \leq n - k$.

Proof: Suppose to the contrary that G contains an independent set $S = \{u_1, u_2, \ldots, u_{n-k+1}\}$ of order $n - k + 1$. Let F be a perfect matching containing the edge u_1v_1. Consider the graph

$$G' = G[V(G) \setminus \{u_1, u_2, \ldots, u_{n-k+1}, v_1, v_2, \ldots, v_{n-k+1}\}],$$

where $u_iv_i \in F$, $1 \le i \le n - k + 1$. Clearly,

$$M = F \setminus \{u_1, u_2, \ldots, u_{n-k+1}, v_1, v_2, \ldots, v_{n-k+1}\}$$

is a perfect matching in G' and $|M| = k - 1$.

If $e = v_iv_j \in E(G)$ for some $1 \le i \ne j \le n - k + 1$, then $M \cup \{e\}$ is a matching of size k in G. Further, as $G - V(M \cup \{e\})$ is a graph on $2n - 2k$ vertices containing the independent set S of order $n - k + 1$, this matching does not extend to a perfect matching in G, a contradiction. Consequently, $\{v_1, v_2, \ldots, v_{n-k+1}\}$ is an independent set.

Now consider the graph

$$G'' = G[V(M) \cup \{v_1, v_2\}].$$

Observe that $G - V(G'')$ is a graph on $2n - 2k$ vertices having S as an independent set of order $n - k + 1$ and thus cannot contain a perfect matching. Consequently, G" cannot have a matching of size k. As $\nu(G'') = 2k$, Theorem 2 implies that $o(G'' - S'') > |S''|$ for some $S'' \subset V(G'')$. In fact, as $|S''|$ and $o(G'' - S'')$ have the same parity we have:

$$o(G'' - S'') \ge |S''| + 2. \tag{1}$$

Now since G' contains a perfect matching, we have $o(G' - S') \le |S'|$ for every $S' \subset V(G')$. If $v_1 \in S''$, then

$$\begin{aligned} o(G'' - S'') &\le o(G' - (S'' \setminus \{v_1\})) + 1 \\ &\le |S''| - 1 + 1 = |S''|, \end{aligned}$$

a contradiction. Hence, $v_1 \notin S''$ and similarly $v_2 \notin S''$. Thus, $S'' \subset V(G')$ and hence

$$o(G' - S'') \le |S''|. \tag{2}$$

Now $o(G'' - S'') \le o(G' - S'') + 2 \le |S''| + 2$ and so, by (1),

$$o(G'' - S'') = |S''| + 2. \tag{3}$$

Further, $|S''| + 2 = o(G'' - S'') \le o(G' - S'') + 2$ and so by (2)

$$|S''| = o(G' - S''). \tag{4}$$

Let w be a vertex in an odd component of $G' - S''$. Then, by (3), $v_i w \notin E(G)$, for $i = 1, 2$. Moreover, v_1 and v_2 are in different components of $G'' - S''$. Further, v_1 and v_2 cannot be joined to the same vertices in even components of $G' - S''$. Now noting that $\{v_1, v_2, \ldots, v_{n-k+1}\}$ is an independent set we have

$$\begin{aligned} d_G(v_1) + d_G(v_2) &\le 2(n - k + 1) + 2|S''| + (2(k - 1) - |S''| \\ &\quad -o(G' - S'')) \\ &= 2n + |S''| - o(G' - S'') \\ &= 2n \qquad \text{(by (4))}. \end{aligned}$$

But since $\delta(G) \ge 2k + 1$, we have $4k + 2 \le 2n$ and so $k < \frac{n}{2}$, a contradiction to the hypothesis of the lemma. This completes the proof. □

Remark 1: The graph $K_{2k} \vee (n - k)K_2$ is a k-extendable graph with minimum degree $2k + 1$ containing an independent set of order $n - k$. Thus, the upper bound on α of Lemma 4 is best possible. Further, the graph $K_{2k+1,2k+1}$ is k-extendable with minimum degree $n = 2k + 1$ containing an independent set of order $n = 2k + 1 > k + 1 = n - k$ for all $k \ge 1$. Thus, the lower bound on k is also best possible.

The following lemma establishes a sufficient condition for a graph G with $\delta(G) \geq n + k - 1$, $1 \leq k \leq n - 2$, and $n - k$ even to be k-extendable.

Lemma 5: Let G be a graph on $2n$ vertices with $\delta(G) \geq n + k - 1$, $1 \leq k \leq n - 2$, and $n - k$ even. If $\alpha(G) \leq n - k$, then G is k-extendable.

Proof: By Theorem 3, G contains a perfect matching. Suppose to the contrary that M is a matching of size k in G that does not extend to a perfect matching. Thus, the graph $G' = G - V(M)$ has no perfect matching. Hence, by Theorem 2, $o(G' - S') > |S'|$ for some $S' \subset V(G')$. Further, since $|S'|$ and $o(G' - S')$ have the same parity, $o(G' - S') \geq |S'| + 2$. Since choosing one vertex from each component of $G' - S'$ yields an independent set of $o(G' - S')$ vertices, $o(G' - S') \leq \alpha(G) \leq n - k$. Thus

$$|S'| + 2 \leq o(G' - S') \leq n - k$$

and then

$$|S'| \leq n - k - 2. \tag{5}$$

Let H be a minimum order odd component of $G' - S'$. Noting that $\delta(G') \geq n - k - 1$ we have for $u \in V(H)$

$$n - k - 1 \leq d_{G'}(u) \leq \nu(H) - 1 + |S'|,$$

and hence

$$\nu(H) \geq n - k - |S'|.$$

Further, by the choice of H,

$$|S'| + \nu(H)\,(o(G' - S')) \leq 2(n - k).$$

Consequently,

$$|S'| + (n - k - |S'|)(|S'| + 2) \le 2\,(n - k).$$

Now if $S' \ne \phi$, then it follows from the above inequality that $|S'| \ge n - k - 1$, contradicting (5). Therefore, $S' = \phi$. Now since $\delta(G') \ge n - k - 1$, G' consists of two odd components each a K_{n-k}. This contradicts the fact that $n - k$ is even, completing the proof. □

Remark 2: The bound on the minimum degree in Lemma 5 is best possible since there exists a graph with minimum degree $n + k - 2$ that has independence number $n - k$ which is not k-extendable. For each k, $1 \le k \le n - 2$, such a graph is $G = \overline{K}_{n-k-1} \vee K_{n+k-2} \vee K_3$ which is drawn in Figure 1. We adopt the convention that a "double line" in our diagram denotes the join between the corresponding graphs.

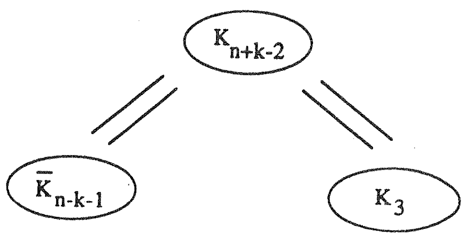

Figure 1

Clearly, G is not k-extendable, since no set of k independent edges of K_{n+k-2} extends to a perfect matching of G. Further, the condition " $n - k$ is even " cannot be dropped since the graph $2K_{n-k} \vee K_{2k}$ has minimum degree $n + k - 1$, independence number 2 and is not k-extendable when $n - k$ is odd.

Lemmas 4 and 5 together yield Theorem 1.

REFERENCES:

[1] J.A. Bondy and U.S.R. Murty, **Graph Theory with Applications,** The Macmillan Press, London, (1976).

[2] M.D. Plummer, **Extending Matchings in Graphs: A Survey,** Discrete Mathematics 127 (1994), 277-292.

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