On complementary path decompositions of the complete multigraph

B. Du

Department of Mathematics
Suzhou University
Suzhou 215006, People's Republic of China

Abstract: We give a complete solution to the existence problem for complementary P_3-decompositions of the complete multigraph, where P_3 denotes the path of length 3.

A complementary decomposition $2\lambda K_v \to (P_3, P_3)$ is an edge decomposition of the complete multigraph λK_v into P_3's with the property that upon taking the complement of each path one obtains a second decomposition of λK_3 into P_3's (where the complement of the path $abcd$ is the path $bdac$). The following result was proven by Granville, Moisiadis and Rees in [1] (and, with a few small exceptions, also follows from the techniques in [3]):

Theorem 1. *There exists a complementary decomposition $2K_v \to (P_3, P_3)$ if and only if $v \equiv 1$ (mod 3).*

In this paper, we give a complete solution to the existence problem for complementary P_3-decompositions of the complete multigraph $2\lambda K_v$. Note that if D is such a decomposition then the set $\{(a,b,c,d) : abcd \in D\}$ is an edge decomposition of $2\lambda K_v$ into K_4's, that is, a $(v,4,2\lambda)$-BIBD. The following result was proven by Hanani in [2]:

Lemma 2. *If there exists a complementary decomposition $2\lambda K_v \to (P_3, P_3)$, then*

$$\lambda(v-1) \equiv 0 \pmod 3$$

and

$$\lambda v(v-1) \equiv 0 \pmod 6.$$

As a consequence of the remarks above and the results in Hanani [2], we need consider only the case $\lambda = 3$.

Australasian Journal of Combinatorics 11(1995), pp.211-213

It is easy to see that the existence of a $(v,4,\lambda)$-BIBD implies the existence of a $2\lambda K_v \to (P_3, P_3)$. From [2] we then have

Lemma 3. *There exists a complementary decomposition* $6K_v \to (P_3, P_3)$ *if* $v \equiv 0, 1$ *(mod 4).*

For our proof, we also need the following initial block constructions.

Lemma 4. *There exists a complementary decomposition* $6K_v \to (P_3, P_3)$ *if* $v \equiv 0, 2$ *(mod 6).*

Proof. In $\mathbb{Z}_{v-1} \cup \{\infty\}$, the required initial blocks are

$$(\infty,\ 0,\ 1,\ 2),$$
$$(0,\ 1,\ \infty,\ 3),$$
$$(0,\ k,\ 2k,\ 3k),\quad 2 \le k \le \tfrac{1}{2}(v-2).$$

Lemma 5. *There exists a complementary decomposition* $6K_q \to (P_3, P_3)$ *if* $q \equiv 3$ *(mod 4) is a prime power.*

Proof. Let $d = \frac{1}{2}(q-1)$, and let x be a generator of $GF(q)$; then the required initial blocks are

$$(x^i,\ x^{i+1},\ x^{i+d},\ x^{i+d+1}),\quad 0 \le i \le d-1.$$

Lemma 6. *There exists a complementary decomposition* $6K_{15} \to (P_3, P_3)$.

Proof. In $\mathbb{Z}_{15}$, the required initial blocks are

(0, 1, 5, 10),
(0, 5, 10, 3),
(0, 12, 14, 8),
(0, 2, 3, 11), two times,
(0, 12, 8, 14), two times.

Theorem 7. *For every integer* $v \ge 4$, *there exists a complmentary decomposition* $6K_v \to (P_3, P_3)$.

Proof. By Hanani [2], it suffices to show that there exists a complementary decomposition for all $v \in \{4, 5, \cdots, 12, 14, 15, 18, 19, 23, 27\}$. If $v \equiv 1$ (mod 3), then Theorem 1 gives the result; if $v \equiv 0, 1$ (mod 4), then Lemma 3 gives the result; if $v \equiv 0, 2$ (mod 6), then Lemma 4 gives the result; if $v \equiv 3$ (mod 4) is a prime power, then Lemma 5 gives the result; for the remaining case $v = 15$, Lemma 6 gives the result. The proof is now complete.

Theorem 8. *There exists a complementary decomposition* $2\lambda K_v \to (P_3, P_3)$ *if and only if*

$$\lambda(v-1) \equiv 0 \pmod 3$$

and

$$\lambda v(v-1) \equiv 0 \pmod 6.$$

Proof. That these conditions are necessary follows from Lemma 2. We need consider only values of λ which are factors of 3, because if $\lambda_1 \mid \lambda_2$ then the existence of a $2\lambda_1 K_v \to (P_3, P_3)$ implies the existence of a $2\lambda_2 K_v \to (P_3, P_3)$. Thus we have the following cases:

$$\lambda = 1 \qquad v \equiv 1 \pmod 3,$$
$$\lambda = 3 \qquad \text{all } v \geq 4.$$

In Theorems 1 and 7 we have established the existence of the required designs.

References

[1] A. Granville, A. Moisiadis and R. Rees, *On complementary decompositions of the complete graph*, Graphs and Combinatorics **5**(1989) 57–61.

[2] H. Hanani, *Balanced incomplete block designs and related designs*, Discrete Math. **11**(1975) 255–369.

[3] R. Rees and C.A. Rodger, *Subdesigns in complementary path decompositions and incomplete two-fold designs with block size four*, Ars Combinatoria **35**(1993), 117–122.

(Received 11/10/94)