Cocyclic Hadamard matrices over $\mathbf{Z}_t \times \mathbf{Z}_2^2$

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Abstract

A natural starting point in a systematic search for cocyclic Hadamard matrices is the study of the case of cocycles over the groups $\mathbf{Z}_t \times \mathbf{Z}_2^2$, for t odd. The solution set includes all Williamson Hadamard matrices, so this set of groups is potentially a uniform source for generation of Hadamard matrices.

We present our analytical and computational results.

1 Introduction

A matrix M of entries ± 1 is termed *group developed* over a finite group G if it is derived from a multiplication table of G by the following method: the ordered elements of G form an index set for M and there is a set mapping $g : G \longrightarrow \{\pm 1\}$ for which $M = [g(ab)],\ a, b \in G$. It will therefore have the same number of occurrences of $+1$ in every row. A Hadamard matrix with the latter property must have side a perfect square $4t^2$ (see [16]).

The simplest groups G over which to search for group developed Hadamard matrices are the cyclic groups $\mathbf{Z}_{4t^2}$ of order $4t^2$. For example, the rows of a group developed matrix over $\mathbf{Z}_4$ are found by circulating the top row, say $g(1), g(a), g(a^2), g(a^3)$, successively one place (to the left). A resultant Hadamard matrix is

$$\begin{bmatrix} 1 & 1 & 1 & - \\ 1 & 1 & - & 1 \\ 1 & - & 1 & 1 \\ - & 1 & 1 & 1 \end{bmatrix}.$$

However, 4 appears to be the only order for which this construction is possible. The *Circulant Hadamard Conjecture* proposes that no circulant Hadamard matrix exists

[1]This paper was presented at the Hadamard Centenary Conference at Wollongong, NSW.
[2]This work is supported by the Australian Research Council

for orders greater than 4, and has been confirmed by Jedwab and Lloyd for orders $4t^2, 1 < t < 5000$, apart from 6 undecided cases (see the survey [13]).

A recent generalisation of group development, termed *cocyclic development*, shows promise as a computational technique for generating Hadamard matrices without the order restriction imposed by group development. The basic theory of cocyclic development of designs is established by Horadam and de Launey in [5, 10, 11]. It links two very active areas of mathematics: combinatorial design theory and group cohomology. New theoretical advances by de Launey [4] and Flannery [8] are also presented to these proceedings.

In this paper, we investigate the existence of cocyclic Hadamard matrices over the groups $G = \mathbf{Z}_t \times \mathbf{Z}_2^2$, for t odd, the next simplest class of groups for our purposes, after the cyclic groups. The Williamson Hadamard matrices are examples of these cocyclic matrices, and we determine the relationship between the two classes of Hadamard matrices. We enumerate the numbers of each for small orders.

The next section reviews the requisite theory of cocyclic matrices over an arbitrary finite group G. In §3 the case $G = \mathbf{Z}_t \times \mathbf{Z}_2^2$, for t odd, is analysed and its relationship with the Williamson matrices determined. The final section lists the results of our computations and describes some open problems and conjectures.

2 Cocyclic binary matrices

This section outlines the theory of cocyclic *binary* matrices (that is, matrices with entries ± 1) established in [5, 10, 11]. We will assume throughout that G is a finite group (multiplicatively written with identity 1) of order v, with elements listed for indexing purposes in a fixed order $G = \{1 = a_1, a_2, \ldots, a_v\}$. $\mathbf{Z}_2$ denotes the set $\{\pm 1\}$, considered as the cyclic group of order 2.

Definition 2.1 A *cocycle (over G)* is a set mapping $f : G \times G \to \mathbf{Z}_2$ which satisfies

$$f(a,b)f(ab,c) = f(a,bc)f(b,c), \qquad \forall a,b,c \in G. \tag{1}$$

A cocycle is *normalised* if

$$f(1,1) = 1. \tag{2}$$

A $v \times v$ binary matrix M is *cocyclic (over G, developed by f)* if there exists a group development function $g : G \to \mathbf{Z}_2$ and a cocycle f such that

$$M = [f(a,b)g(ab)], \quad \forall a,b \in G. \tag{3}$$

If $g \equiv 1$ in (3); that is, $M = [f(a,b)]$, then M is termed *pure* cocyclic. □

A useful consequence of (1) is that

$$f(1,1) = f(a,1) = f(1,a), \quad \forall a \in G. \tag{4}$$

The term "cocycle" derives from group cohomology: a function satisfying (1) is a 2-dimensional cocycle of the unnormalised standard complex for computing the cohomology of the group G with trivial coefficients in $\mathbf{Z}_2$ [2, pp. 92-93].

Three types of cocyclic matrices are important for the results which follow.

Clearly, group developed binary matrices are cocyclic, developed by $\mathbf{1}_v$, the *trivial* cocycle which takes each (a, b) to 1. However, there is a more fundamental relationship between group development functions and cocycles: it is easy to check that each $g : G \to \mathbf{Z}_2$ determines a cocycle.

Definition 2.2 A cocycle f over G is *principal* (or a *coboundary*) if there exists a set mapping $g : G \to \mathbf{Z}_2$ such that $f(a, b) = g(a)g(b)g(ab)^{-1}$ for all a, b in G. □

The pure principal cocyclic binary matrices which result are essential components of the structure of an arbitrary pure cocyclic matrix.

A second essential component of the structure of an arbitrary pure cocyclic matrix is the back negacyclic binary matrix.

Definition 2.3 [10, 4.5] *The cocycle* $w_t : \mathbf{Z}_t \times \mathbf{Z}_t \to \mathbf{Z}_2$ *defined by* $w_t(a^i, a^j) = (-1)^{\lfloor (i+j)/t \rfloor}$, $0 \leq i, j < t$, *determines the back negacyclic matrix of order* t*:*

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & - \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & - & - \\ 1 & - & \cdots & - & - \end{bmatrix}. \quad \square$$

Finally, in the general theory of cocyclic combinatorial designs described in [10], the Williamson array

$$\begin{bmatrix} w & x & y & z \\ x & -w & z & -y \\ y & -z & -w & x \\ z & y & -x & -w \end{bmatrix} \tag{5}$$

is a cocyclic orthogonal design over the group $\mathbf{Z}_2^2$, provided the indeterminates w, x, y, z all commute (see [5, Table 1(c)]). When they are all set equal to 1, a cocyclic Hadamard matrix results.

The following consequence of (1) permits us to show that two standard constructions of Hadamard matrices preserve cocyclic matrices.

Proposition 2.4 (cf [5, 3.2.i]) *If* f *is a cocycle over* G *and* p *is a cocycle over* H *then* $f \times p$*, defined by*

$$f \times p\ ((a, b), (c, d)) = f(a, c)\ p(b, d), \quad a, c \in G, \quad b, d \in H,$$

is a cocycle over $G \times H$. □

The corresponding pure cocyclic matrix is the Kronecker (tensor) product of the two pure cocyclic factors:

$$[f \times p\ ((a,b),(c,d))] = [f(a,c)] \otimes [p(b,d)].$$

Since the Kronecker product of Hadamard matrices is Hadamard, the Kronecker product of cocyclic Hadamard matrices is cocyclic Hadamard. When this result is iteratively applied to the back negacyclic matrix of order two, which is a Hadamard matrix, we obtain the cocyclic version of a standard construction for Hadamard matrices.

Corollary 2.5 [5, 3.3] *If there exists a cocyclic Hadamard matrix of order v over G then, for all integers $s > 0$, there exists a cocyclic Hadamard matrix of order $2^s v$ over $\mathbf{Z}_2^s \times G$.* □

Obviously, a necessary condition for any binary matrix to be a Hadamard matrix is that the dot product of the first row with each other row, is 0. For cocyclic matrices, this is in fact sufficient.

Lemma 2.6 *A cocyclic binary matrix is a Hadamard matrix if and only if the dot product of the first row with each other row is 0.*

Proof. Let $[f(a,b)g(ab)]$ be a cocyclic matrix, choose distinct elements $a, c \in G$ and let $d = ca^{-1} \neq 1$. Then the dot product of the first row with the row indexed by d is

$$P = \sum_{b \in G} f(1,b)g(b)f(d,b)g(db) = \pm f(ca^{-1},a) \sum_{b \in G} f(ca^{-1},ab)f(1,ab)g(ab)g(cb)$$

on replacing b by ab, by (4). Thus

$$P = \pm \sum_{b \in G} f(ca^{-1},a)f(ca^{-1},ab)g(ab)g(cb) = \pm \sum_{b \in G} f(a,b)g(ab)f(c,b)g(cb),$$

since by (1), $f(ca^{-1},a)f(ca^{-1},ab)^{-1} = f(a,b)f(c,b)^{-1}$ and each element in $\mathbf{Z}_2$ is its own inverse. The result follows by definition. □

Thus there is a computationally cheap test to check if a cocyclic binary matrix is Hadamard or not, requiring at most $(v-1)^2$ binary additions.

Two binary matrices are *Hadamard equivalent*, denoted by $\sim_h$, if one can be obtained from the other by a sequence of row or column permutations or negations.

On multiplying the a^{th} row (b^{th} column) of $[f(a,b)g(ab)]$ by $g(a)^{-1}$ ($g(b)^{-1}$),

$$[f(a,b)g(ab)] \sim_h [g(a)^{-1}g(b)^{-1}g(ab)f(a,b)],$$

so by Definition (2.2), each cocyclic matrix is Hadamard equivalent to a pure cocyclic matrix [3]. Every binary matrix is Hadamard equivalent to at least one *normalised*

[3] This equivalence justifies the common usage of "cocyclic" to mean "pure cocyclic"

binary matrix, having first row and column entries all +1. It follows from (4) that each cocyclic matrix is Hadamard equivalent to a normalised pure cocyclic matrix.

Hadamard equivalence is an equivalence relation on Hadamard matrices, so the search for cocyclic Hadamard matrices can be restricted to the normalised pure cocyclic matrices.

In [10, 11] we specify the structure of a normalised pure cocyclic matrix to be a Hadamard product

$$S \bullet P \bullet C, \tag{6}$$

where S, a Symmetric matrix, is a Kronecker product of an all 1's matrix and back negacyclic matrices, P is developed by a Principal cocycle, so its symmetry properties reflect symmetries in the multiplication table of G, and C is developed by a "Commutator" cocycle, determined by the Schur multiplicator $H_2(G)$ of G. When G is abelian, the entries in the Hadamard factors (6) can be derived directly from the generating set (see [11, 4.2]).

3 Cocyclic Hadamard matrices over $\mathbf{Z}_t \times \mathbf{Z}_2^2$

The case $G = \mathbf{Z}_t \times \mathbf{Z}_2^2$ is a natural starting point in the search for cocyclic Hadamard matrices, for several reasons. Firstly, such groups exist for every order a multiple of 4. Secondly, the factorisation (6) is well understood for abelian groups. The most telling reason, however, is that each Williamson Hadamard matrix of order $4t$ is (Hadamard equivalent to) a cocyclic matrix over $\mathbf{Z}_t \times \mathbf{Z}_2^2$. The Williamson Hadamard matrices have been extensively studied for 50 years, since the appearance of Williamson's paper [18].

We begin by specifying the Hadamard factors (6) of a normalised pure cocyclic matrix over $\mathbf{Z}_t \times \mathbf{Z}_2^2$, for t *odd.* Represent this primary decomposition of G by $< x : x^t > \times < u, v : u^2, v^2, [u, v] >$ and fix the ordering to be:

$$(x^i, 1) < (x^i, u) < (x^i, v) < (x^i, uv), \quad 0 \leq i < t,$$

$$(x^i, uv) < (x^{i+1}, 1), \quad 0 \leq i < t - 1.$$

The formula [11, 4.2], applied to the isomorphic torsion invariant form $< u : u^2 = 1 > \times < y : y^{2t} = 1 >$, $y = vx$ of G, states that any cocycle is uniquely determined by its action on the $4t - 1$ elements (u, u), $(y, y^i), (u, y^i), 1 \leq i < 2t$, of infinite order, and the single "commutator" element $C(u, y) = (u, y)(y, u)^{-1}$ of order 2, in a subquotient of $\mathbf{Z}(G \times G)$. (The action on any other (a, b) is found by iterating the cocycle equation (1) to write (a, b) as a product of these generators.) The action on the generators of infinite order is described by the product $S \bullet P$, and on the single generator of order 2, by C.

The symmetric matrix S has the form (see [10, Def. 13.2.i])

$$S = \mathbf{1}_t \otimes \begin{bmatrix} 1 & 1 \\ 1 & B \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & A \end{bmatrix} = \mathbf{1}_t \otimes \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & A & 1 & A \\ 1 & 1 & B & B \\ 1 & A & B & AB \end{bmatrix} \qquad A, B = \pm 1.$$

The sole nontrivial commutator cocycle acts trivially on all the generators of infinite order and maps $(u,y)(y,u)^{-1}$ to -1. In the lemma following we simplify this action to derive the possible C.

Lemma 3.1 *For $G = \mathbf{Z}_t \times \mathbf{Z}_2^2$,*

$$C = \mathbf{1}_t \otimes \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & K & 1 & K \\ 1 & K & 1 & K \end{bmatrix} \qquad K = \pm 1.$$

Proof. In the notation above, note

$$< y : y^{2t} = 1 > = < v : v^2 = 1 > \times < x : x^t = 1 > .$$

Then $u = u^1(vx)^0, v = u^0(vx)^t$ and $x = u^0(vx)^{t+1}$ in G, and by [5, 4.4.ii], $C(u,y) = C(u,v)C(u,x)$. But by [5, 4.4.iii], $C(u,x) = C(u,y)^{1.(t+1)-0.0} \equiv 1$, since $C(u,y)$ has order 2. Thus $C(u,y) = C(u,v)$, and the sole nontrivial commutator cocycle acts trivially on all the generators of infinite order and maps $(u,v)(v,u)^{-1}$ to -1. That is, it is the product $\mathbf{1}_t \times f$ derived from the commutator cocycle f over $\mathbf{Z}_2^2$ given in [5, Table 1(b)] (see also [10, 13.4.i]). □

The matrix P developed by principal cocycles is found simply by normalising an arbitrary group developed matrix over $\mathbf{Z}_t \times \mathbf{Z}_2^2$, and so may be replaced in (6) by a Hadamard equivalent group developed matrix U. This will be a $t \times t$ block-backcirculant matrix with top row $U_1, U_2, \ldots, U_t$ consisting of 4×4 block matrices each group developed over $\mathbf{Z}_2^2$. Denote the top row of U_i by n_i, x_i, y_i, z_i.

We conclude that each normalised pure cocyclic matrix over $\mathbf{Z}_t \times \mathbf{Z}_2^2$ is $\sim_h$ to a $t \times t$ block-backcirculant matrix W with top row $W_1, W_2, \ldots, W_t$, where

$$W_i = \begin{bmatrix} n_i & x_i & y_i & z_i \\ x_i & An_i & z_i & Ay_i \\ y_i & Kz_i & Bn_i & BKx_i \\ z_i & AKy_i & Bx_i & ABKn_i \end{bmatrix}, \quad 1 \le i \le t. \qquad (7)$$

The commutator variable K appears to play the most significant role in the behaviour of W. Clearly, W is a symmetric matrix if and only if $K = 1$. If also $A = B = 1$, W is $\sim_h$ to a group developed matrix and, if it is Hadamard then t must be a perfect square.

It remains to link cocyclic Hadamard matrices to Williamson Hadamard matrices.

Definition 3.2 A *Williamson* Hadamard matrix is a Hadamard matrix of the form (5) in which each of w, x, y, z is a circulant symmetric binary $t \times t$ matrix. (In the usual definition, the last three rows of (5) are multiplied by -1.)

Because they are circulant, the blocks w, x, y, z commute in pairs and, by symmetry, must have first rows of the (almost) symmetric form $x_0, x_1, \ldots x_{t-1}$ in which $x_{t-i} = x_i$, $1 \leq i \leq t-1$. Turyn [15] proved that Williamson Hadamard matrices exist for $4t = 2(q+1)$ for any prime power $q \equiv 1 \pmod 4$. Lists of solutions for the range $t = 3, \ldots, 27, 37, 43$ are published in [1, 9, 16, 17], and existence results are claimed for the values $t = 29, 31, 33, 39$ in [6, 7]. However, there are errors in [9, 16] in particular, which are detailed in §4 below.

If each circulant $t \times t$ block in a Williamson Hadamard matrix is reflected in its central column and the columns are cycled once to the right, a backcirculant $t \times t$ matrix with unchanged initial row is produced. We have the following result.

Lemma 3.3 *A Williamson Hadamard matrix is* $\sim_h$ *to a Hadamard matrix of the form* (5) *in which each of* w, x, y, z *is a backcirculant matrix with first row of the form* $x_0, x_1, \ldots x_{t-1}$ *in which* $x_{t-i} = x_i$, $1 \leq i \leq t-1$.

Re-ordering rows and columns of the matrix W with blocks (7) according to $\mathbf{Z}_2^2 \times \mathbf{Z}_t$, rather than $\mathbf{Z}_t \times \mathbf{Z}_2^2$, produces a Hadamard equivalent matrix of the form

$$\begin{bmatrix} N & X & Y & Z \\ X & AN & Z & AY \\ Y & KZ & BN & BKX \\ Z & AKY & BX & ABKN \end{bmatrix}, \tag{8}$$

where N, X, Y, Z are $t \times t$ backcirculant matrices with top rows $n_1, \ldots, n_t$, $x_1, \ldots, x_t$, $y_1 \ldots, y_t$, $z_1, \ldots, z_t$, respectively. This is clearly a generalisation of the Williamson array (5), and is technically cocyclic over $\mathbf{Z}_2^2 \times \mathbf{Z}_t$, with the Kronecker products in S and C of (6) taken in reverse order.

Putting the previous remarks together with orthogonality leads to the next result.

Lemma 3.4 *A Williamson Hadamard matrix is* $\sim_h$ *to a cocyclic Hadamard matrix* W *of the form* (7) *over* $\mathbf{Z}_t \times \mathbf{Z}_2^2$, *in which* $A = B = K = -1$ *and* $W_{i+1} = W_{t-i+1}, 1 \leq i \leq t-1$.

An immediate question to ask is: does every cocyclic Hadamard matrix over $\mathbf{Z}_t \times \mathbf{Z}_2^2$ have this precise "Williamson" form? Of course, once Hadamard equivalence is considered, we would not expect the symmetry condition on top blocks to survive. For example, cycling the columns of blocks in W preserves the cocyclic structure while destroying symmetry. Certainly there are more cocyclic Hadamard matrices than those of "Williamson" form, if no account of equivalence is taken (see the table in §4 below). We can also show (though no details will be given here) that if $K = -1$ then $A = B = -1$. Thus the question is better phrased as :

are there any symmetric cocyclic Hadamard matrices over $\mathbf{Z}_t \times \mathbf{Z}_2^2$; that is, with $K = 1$.

For some values of t, the answer is no.

Lemma 3.5 *Suppose W in (7) is a Hadamard matrix. Then $4t$ is a sum of four odd squares. If $K = 1$ then t is a sum of two squares; that is, $t \equiv 1 \pmod 4$.*

Proof. Write a, b, c, d for the respective number of n_i, x_i, y_i, z_i in the top row of W which are -1. Summing the dot product of the 1st row with the first row of each block in (7) in two different ways gives the equation

$$4t = (t-2a)^2 + (t-2b)^2 + (t-2c)^2 + (t-2d)^2 \tag{9}$$

and with the 2nd, 3rd and 4th rows of each block gives

$$\begin{aligned} 0 &= (1+A)\left[(t-2a)(t-2b) + (t-2c)(t-2d)\right] \\ 0 &= (1+B)\left[(t-2a)(t-2c) + K(t-2b)(t-2d)\right] \\ 0 &= (1+ABK)\left[(t-2a)(t-2d) + B(t-2b)(t-2c)\right], \end{aligned} \tag{10}$$

The first of these (9) is the familiar "four odd squares" condition [18] necessary for the existence of Williamson Hadamard matrices. We know that if $K = -1$ then $A = B = -1$, so the remaining equations are automatically satisfied. If $K = 1$ and the four equations above are summed, the sum reduces to

$$t = \begin{cases} (2t-a-b-c-d)^2 & \text{A = B = K = 1} \\ (t-a-b)^2 + (t-c-d)^2 & \text{A = K = 1, B = -1} \\ (t-a-c)^2 + (t-b-d)^2 & \text{A = -1, B = K = 1} \\ (t-a-d)^2 + (b-c)^2 & \text{A = B = -1, K = 1} \end{cases} \tag{11}$$

The result follows. □

Thus no symmetric Hadamard cocyclic matrix developed over $\mathbf{Z}_t \times \mathbf{Z}_2^2$ exists for $t \equiv 3 \pmod 4$. Further restrictions on the existence of such a matrix can be deduced if $A = B = 1$, since then t is a perfect square and the three equations (11) above imply $(t-2a)^2 = (t-2b)^2 = (t-2c)^2 = (t-2d)^2$. Substitution into the first equation of (11) shows that $a = \frac{1}{2}\,(t \pm \sqrt{t})$, that three of $\{a, b, c, d\}$ are equal (to x, say) and the fourth element is $t - x$.

Lemma 3.6 *With the notation above, suppose W in (7) is a Hadamard matrix with $A = B = K = 1$. Then three of the elements $\{a, b, c, d\}$ are equal to one of the terms $\frac{1}{2}(t \pm \sqrt{t})$ and the fourth equals the other.*

The number of potential group developed cocyclic Hadamard matrices is thus very small; namely $8\binom{t}{x}^4$, where t is a perfect square and $x = \frac{1}{2}(t - \sqrt{t})$.

4 Computations

For ease of computation it has proved preferable to work with a partly normalised version of W of (7). On multiplying the first column of every block by n_1, the second column by x_1, the third column by y_1 and the fourth column by $n_1x_1y_1$, and then on multiplying the second row of every block by n_1x_1, the third row by n_1y_1 and the fourth row by x_1y_1, we obtain a Hadamard equivalent $t \times t$ block-backcirculant matrix V with top row $V_1, V_2, \ldots, V_t$, where V_i has the form

$$V_i = m_i \begin{bmatrix} 1 & X_i & Y_i & Z_i \\ X_i & A & Z_i & AY_i \\ Y_i & KZ_i & B & BKX_i \\ Z_i & AKY_i & BX_i & ABK \end{bmatrix}, \quad 1 \leq i \leq t, \tag{12}$$

and $m_1 = X_1 = Y_1 = 1$.

There are $4t$ binary variables in V, of which A and B derive from negacyclic matrices, K corresponds to a commutator cocycle, and the remaining $4t-3$ correspond to principal cocycles. We introduce notation to simplify our analysis of these matrices.

Definition 4.1 A matrix of the form (12) will be denoted (setting $D = Z_1$)

$$\{A, B, K \mid \vec{M}; \vec{X}; \vec{Y}; D, \vec{Z}\},$$

where $\vec{M} = m_2, \ldots, m_t$, $\vec{X} = X_2, \ldots, X_t$, $\vec{Y} = Y_2, \ldots, Y_t$, and $\vec{Z} = Z_2, \ldots, Z_t$. (Recall that $m_1 = X_1 = Y_1 = 1$.) The set of 2^{4t} such cocyclic matrices will be denoted $\mathcal{C}_t$, the subset of Hadamard matrices in $\mathcal{C}_t$ will be denoted $\mathcal{H}_t$, and the subset of Williamson Hadamard matrices in $\mathcal{H}_t$ will be denoted $\mathcal{W}_t$. □

Programs have been implemented to list $\mathcal{H}_t$ and $\mathcal{W}_t$, and summary results are tabulated below. For each t, the number of Williamson and cocyclic Hadamard matrices of the form (12), where known, is listed, together with an example of the former in format $\vec{M}; \vec{X}; \vec{Y}; D, \vec{Z}$. This last is provided for reference purposes, as some previously published lists [9, 16] contain errors (the only correct cases in [9] are for $t = 23, 25$, while in [16] the cases $t = 11, 15, 17$ are incorrect).

Except for the initial case $t = 1$, $K = -1$ for every element of $\mathcal{H}_t$, (and hence $A = B = -1$). We do not yet know why no symmetric cocyclic Hadamard matrices over $\mathbf{Z}_t \times \mathbf{Z}_2^2$ have been found for the orders $t \equiv 1 \pmod 4$, but de Launey has suggested that an argument of [12] might apply. We propose the following analogue of the Circulant Hadamard Conjecture.

Conjecture 4.2 *There are no symmetric cocyclic Hadamard matrices over $\mathbf{Z}_t \times \mathbf{Z}_2^2$ for odd $t > 1$.*

Of course, even if the conjecture is true, there are other groups of order $4t$ over which to search for symmetric cocyclic Hadamard matrices.

t	$\#\mathcal{W}_t$	$\#\mathcal{H}_t$	*Williamson Hadamard matrix*
1	1	6	
3	8	24	− −; − −; + +; + + +
5	24	120	− − − −; − + + −; + − − +; + + + + +
7	120	840	+ − + + − +; − − + + − −; + + − − + +; + + + + + + +
9	264	3240	− + + − − + + − − − − + + − − − + + + − − + + + + + + + + + + + +
11	240		− + + + − − + + + − + − + − + + − + − + − + + − + + − + + − − − − + + + + + + − −
13	648		+ − − − + − − + − − − + − + + − − − − − − + + − + − − + + + + + + − − + − − − − − − − − − − − − −
15	576		+ − − + − − − − − − + − − + + + − − + − − − − + − − + + − − + + − + + + + − + + − − − − − − − − − − − − − − − − −
17			− + − − + + − − − − + + − − + − − − − − + − + + + + − + − − − − + − − + + − + + + + − + + − − + − − + + − + + + + + + + + − + + −
19			− + − − + − − + + + + − − + − − + − − + − + + − − − − − − − − + + − + − + − + − − + + + + + + + + − − + − + − − − − − − − − − − − − − − − − − − −
21			− − − + + + + + − + + − + + + + + − − − + − + − − + + − − − − − − + + − − + − + − + − + + − − + + + + + + − − + + − + −
23			+ − − − + − − − + − + + − + − − − + − − − + − − − + + − + + + − + + − + + + − + + − − − + − + + − − − + + + + + + + + − − − + + − + − − + − + − + − + + − + + − + + − + − + − + −
25			− − + + − + − + − − − − − − − − + − + − + + − − + − + − − + − − − − + + + + − − − − + − − + − + − − − + − + + − + − + − − + − + − + + − + − − − − − + − − + + − + − + + − − + + − + − + + − − + −

Acknowledgment:

The authors are grateful to Mr. Greg Gamble for his assistance with the computational aspects of this paper.

References

[1] L. D. Baumert, M. Hall, Jr, Hadamard matrices of Williamson type, Math. Comput. 19 (1965), 442 - 447.

[2] K.S. Brown, *Cohomology of groups*, Graduate Texts in Math. 87, Springer-Verlag, New York, 1982.

[3] W. de Launey, On the construction of n-dimensional designs from 2-dimensional designs, Australas. J. Combin. 1 (1990), 67-81.

[4] W. de Launey, Cocyclic Hadamard matrices and relative difference sets, seminar notes, RMIT Mathematics Department, November 1993; Hadamard Centenary Conference, U. of Wollongong, December 1993.

[5] W. de Launey and K. J. Horadam, A weak difference set construction for higher-dimensional designs, Designs, Codes and Cryptography 3 (1993), 75-87.

[6] D. Z. Dokovic, Williamson matrices of orders 4.29 and 4.31, J. Combin. Theory Ser. A, 59 (1992), 309 - 311.

[7] D. Z. Dokovic, Williamson matrices of orders $4n$ for $n = 33, 35, 39$, Disc. Math. 115 (1993) 267-271.

[8] D. Flannery, Calculation of cocyclic matrices, Hadamard Centenary Conference, University of Wollongong, December 1993.

[9] A. Hedayat and W. D. Wallis, Hadamard matrices and their applications, Ann. Statist. 6, (1978) 1184-1238.

[10] K. J. Horadam and W. de Launey, Cocyclic development of designs, J. Algebraic Combinatorics 2 (1993) 267-290, Erratum 3 (1994) 129.

[11] K. J. Horadam and W. de Launey, Generation of cocyclic Hadamard matrices, Research Report No. 2, Mathematics Department, RMIT, March 1993.

[12] J. Jedwab, Perfect arrays, Barker arrays and difference sets, Ph. D. thesis, Royal Holloway and Bedford New College, University of London, 1991.

[13] C. Lin and W. D. Wallis, Barker sequences and circulant Hadamard matrices, J. Combin. Inform. System Sci., 18(1993),19-25.

[14] J. Seberry, M. Yamada, Hadamard matrices, sequences and block designs, in Contemporary Design Theory, eds J. H. Dinitz and D. R. Stinson, John Wiley & Sons (1992).

[15] R. J. Turyn, An infinite class of Williamson matrices, J. Combin. Theory Ser. A, 12 (1972), 319 - 321.

[16] W. D. Wallis, Combinatorial designs, Marcel Dekker, New York, 1988.

[17] W.D. Wallis, A.P. Street and J.S. Wallis, *Combinatorics: Room Squares, sum-free sets, Hadamard Matrices*, Lecture Notes in Math. 292, Springer-Verlag, Berlin, 1972.

[18] Williamson, J., Hadamard's determinant theorem and the sum of four squares, Duke Math. J. 11 (1944), 65-81.

(Received 22/2/94)