Fibonacci numbers, ordered partitions, and transformations of a finite set.

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A method of representing transformations of a finite set pictorially is described. These pictures of a function are used to count certain idempotent transformations, and interesting formulae for the Fibonacci numbers are obtained.

A *transformation*, θ, of $\{1, 2, \ldots, n\}$ is a function from the set to itself, and may be realized in $\mathbf{R}^3$ as a *vine*, consisting of:

a set $\{(1,0,1), (2,0,1), \ldots, (n,0,1)\}$ of *initial nodes;*

a set $\{(1,0,0), (2,0,0), \ldots, (n,0,0)\}$ of *terminal nodes;*

for each $i \in \{1, 2, \ldots, n\}$ a string from $(i, 0, 1)$ to $(\theta(i), 0, 0)$.

No part of any string extends beyond the region $0 \leq z \leq 1$, and if two strings intersect then they join together, from the point of intersection, to meet the same terminal node.

Two vines are considered equivalent if one can be continuously deformed to the other, so that at each stage of the deformation, the strings form a vine.

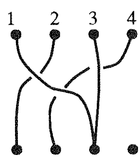

Figure 1: A vine representing $(1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 3, 4 \mapsto 2)$.

To compose the vines u and v, shrink them so that u lies in the region $\frac{1}{2} \leq z \leq 1$ and v is contained in $0 \leq z \leq \frac{1}{2}$. Identify the terminal nodes of u with the initial nodes of v. The j^{th} string of uv is obtained by following the j^{th} string of u until it meets a string of v and then following this string to a terminal node of v. With this composition, the equivalence classes of vines with n strings form a monoid, studied more fully in [2].

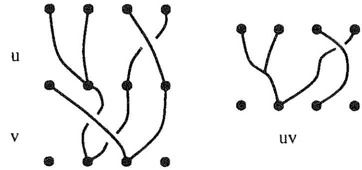

Figure 2: The composition process.

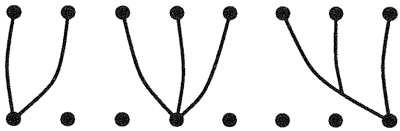

Figure 3: A simple vine.

A vine will be called *simple* if no two of its strings cross. The simple vines form a monoid isomorphic to $\mathcal{O}_n$, the monoid of order-preserving transformations of $\{1, 2, \ldots, n\}$. Each simple vine is determined by an ordered partition of n and an r-subset of $\{1, 2, \ldots, n\}$, the *target set.* (An *ordered partition* or *composition* of n is a sequence of positive integers summing to n.) For example, the simple vine in Figure 3 corresponds to the ordered partition (2,3,3) and the subset $\{1, 4, 8\}$. An ordered partition of n into r parts is obtained by placing 'spacers' in $r-1$ of the $n-1$ 'gaps' between the first n positive integers on the number line. Hence the following result in [1] is obtained:

$$
\begin{aligned}
|\mathcal{O}_n| &= \sum_{r=1}^{n} \binom{n-1}{r-1} \binom{n}{r} \\
&= \binom{2n-1}{n-1}
\end{aligned}
$$

using Vandermonde's identity ([3], page 8).

A similar argument counts the transformations in $\mathcal{O}_n$ that fix n: if $\mathcal{O}'_n$ is the monoid of all such functions,

$$
\begin{aligned}
|\mathcal{O}'_n| &= \binom{2n-2}{n-1} \\
&= nC_{n-1}
\end{aligned}
$$

where C_{n-1} is the $(n-1)^{\text{th}}$ *Catalan number* ([3], page 101).

A simple vine, s, is idempotent if and only if the following condition holds for all $i \in \{1, 2, \ldots, n\}$: if any strings of s meet the i^{th} terminal node, then the i^{th} does. So for each ordered partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of n there are $\lambda_1 \times \lambda_2 \times \cdots \times \lambda_r$ idempotent simple vines.

Proposition 1 *Denoting by $\phi(n)$ the number of idempotent, order preserving transformations of $\{1, 2, \ldots, n\}$, and by $\psi(n)$ the number of such functions that fix n;*

(i)

$$\phi(n) = \sum_{\lambda_1+\lambda_2+\cdots+\lambda_r=n} \lambda_1 \times \lambda_2 \times \cdots \times \lambda_r, \tag{1}$$

(ii)

$$\psi(n) = \sum_{\lambda_1+\lambda_2+\cdots+\lambda_r=n} \lambda_1 \times \lambda_2 \times \cdots \times \lambda_{r-1} \times 1. \tag{2}$$

□

If a simple vine v has the last r strings joined together, then for v to be idempotent, these must meet one of the last r terminal nodes. Thus the number of idempotent simple vines with the last r strings joined together is $r\phi(n-r)$, and (see [1]),

$$\phi(n) = \phi(n-1) + 2\phi(n-2) + \cdots + (n-1)\phi(1) + n\phi(0)$$

with $\phi(0) = 1$, $\phi(1) = 1$, and $\phi(2) = 3$. After a little work (see [1]), this recursion formula gives, for $n \geq 1$,

$$\phi(n) = F_{2n} \tag{3}$$

where F_i is the i^{th} Fibonacci number ([3], page 101). Now there are $\phi(n-i)$ idempotent simple vines that take the last i strings to the final node, so

$$\begin{aligned} \psi(n) &= \phi(n-1) + \phi(n-2) + \cdots + \phi(2) + \phi(1) + \phi(0) \\ &= F_{2n-2} + F_{2n-4} + \cdots + F_4 + F_2 + 1 \\ &= F_{2n-2} + F_{2n-4} + \cdots + F_4 + F_3 \text{ (since } 1 = F_1) \\ &\vdots \\ &= F_{2n-1}. \end{aligned} \tag{4}$$

Comparison of (1) with (3) and (2) with (4) gives interesting formulae for the Fibonacci numbers.

Theorem 1 *Let F_k denote the k^{th} Fibonacci number. For $n \geq 1$:*

(i) *(See [4], page 46.)* $F_{2n} = \displaystyle\sum_{\lambda_1+\lambda_2+\cdots+\lambda_r=n} \lambda_1 \times \lambda_2 \times \cdots \times \lambda_r$;

(ii) $F_{2n-1} = \displaystyle\sum_{\lambda_1+\lambda_2+\cdots+\lambda_r=n} \lambda_1 \times \lambda_2 \times \cdots \times \lambda_{r-1} \times 1.$

To extend these results requires some new notation.

Definition 1 For $n \geq 1$ and $0 \leq k \leq n$,

$$\chi^{(k)}(n) = \sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_r=n \\ r \geq k}} \lambda_1 \times \lambda_2 \times \cdots \lambda_{r-k} \times 1^k.$$

Note that $\chi^{(0)}(0) = 0$, and that $\chi^{(k)}(n) = 0$ if $k > n$. Also, for $n \geq 1$, $\phi(n) = \chi^{(0)}(n)$ and $\psi(n) = \chi^{(1)}(n)$.

As an example, consider the case $n = 3$. The contribution each ordered partition makes to $\chi^{(k)}(3)$ is listed for $k = 0, 1, 2, 3$.

Ordered partition	$k=0$	$k=1$	$k=2$	$k=3$
1, 1, 1	1	1	1	1
1, 2	2	1	1	0
2, 1	2	2	1	0
3	3	1	0	0

Adding each column up gives $\chi^{(0)}(3) = 8$, $\chi^{(1)}(3) = 5$, $\chi^{(2)}(3) = 3$, and $\chi^{(3)}(3) = 1$.

Definition 2 A simple vine with corresponding ordered partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ and target set $\{\rho_1, \rho_2, \ldots, \rho_r\}$ is *natural of degree k* if:

(i) $\quad 0 \leq k \leq r$; and,

(ii)

$$\begin{aligned} \rho_r &= n \\ \rho_{r-1} &= n - \lambda_r \\ &\vdots \\ \rho_{r-k+1} &= n - \lambda_r - \lambda_{r-1} - \cdots - \lambda_{r-k+2}. \end{aligned}$$

If a simple vine is natural of degree k, then for $0 \leq l < k$, it is natural of degree l. Of the simple vines with corresponding ordered partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$, none are natural of degree m for $m > r$, and exactly one is natural of degree r.

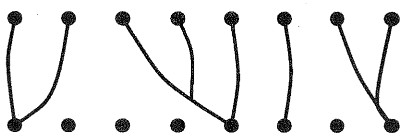

Figure 4: This simple vine, corresponding to $(2, 3, 1, 2)$, is natural of degree 3.

Proposition 2 *The number of idempotent simple n-vines that are natural of degree k is* $\chi^{(k)}(n)$.

Proof: If $r \geq k$, there are

$$\lambda_1 \times \lambda_2 \times \cdots \times \lambda_{r-k} \times 1^k$$

idempotent simple vines, natural of degree k, corresponding to the ordered partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$. If $r < k$, there are no such vines. $\square$

Theorem 2 (*i*) *For* $n \geq 1$, $\chi^{(n)}(n) = 1$.

(*ii*) *For* $1 \leq k < n$,

$$\chi^{(k)}(n) = \chi^{(k)}(n-1) + \chi^{(k-1)}(n-1).$$

(*iii*) *For* $n \geq 1$,

$$\chi^{(0)}(n) = \chi^{(0)}(n-1) + \chi^{(1)}(n).$$

□

Using these formulae, a "Fibonacci triangle" can be constructed recursively:

$$\begin{array}{lllll}
\chi^{(0)}(0) & & & & \\
\chi^{(0)}(1) & \chi^{(1)}(1) & & & \\
\chi^{(0)}(2) & \chi^{(1)}(2) & \chi^{(2)}(2) & & \\
\chi^{(0)}(3) & \chi^{(1)}(3) & \chi^{(2)}(3) & \chi^{(3)}(3) & \\
\chi^{(0)}(4) & \chi^{(1)}(4) & \chi^{(2)}(4) & \chi^{(3)}(4) & \chi^{(4)}(4) \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array} .$$

The entries begin:

$$\begin{array}{ccccccc}
0 & & & & & & \\
1 & 1 & & & & & \\
3 & 2 & 1 & & & & \\
8 & 5 & 3 & 1 & & & \\
21 & 13 & 8 & 4 & 1 & & \\
55 & 34 & 21 & 12 & 5 & 1 & \\
144 & 89 & 55 & 33 & 17 & 6 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array} .$$

Each entry in the triangle differs from the entry one row above and two columns to the left by a binomial coefficient. In fact, an induction using Theorem 2 gives:

Theorem 3 *For* $k \geq 2$ *and* $n \geq k$,

$$\chi^{(k)}(n) = \chi^{(k-2)}(n-1) - \binom{n-2}{k-3}.$$

The solution of this recurrence yields more results in the spirit of Theorem 1.

References

[1] Howie, J. M. *Products of idempotents in certain semigroups of transformations* Proc. Edinburgh Math. Soc. 17 (1971) p. 223-236

[2] Lavers, T. G. *The Vine Monoid.* Research Report 93-30, School of Mathematics and Statistics, University of Sydney.

[3] Riordan, J. *Combinatorial Identities.* John Wiley and Sons, Inc. (1968).

[4] Stanley, R. P. *Enumerative Combinatorics. (Volume I)* Wadsworth and Brooks (1986).

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