FURTHER RESULTS ON SMALLEST DEFINING SETS OF WELL KNOWN DESIGNS

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ABSTRACT

A set of blocks which can be a subset of only one t–(v, k, λ_t) design has been termed a *defining set* of that design. In an earlier paper the author examined the smallest such sets of blocks for certain designs; that work is continued here for further designs. Improved lower bounds for the cardinality of a defining set are given for the affine and projective planes of order n.

1. Defining Sets: definitions and basic results

Any *design* we consider is a collection of b *blocks* (k–subsets) chosen from a set of v elements. The term *block design* refers to a collection of blocks chosen in such a way that every element belongs to exactly r blocks. If $k < v$ we say the block design is *incomplete*. If every subset of t elements belongs to exactly λ_t blocks for some constant λ_t, we call the design a t*–design* and indicate its parameters by t–(v, k, λ_t). When $t \geq 2$ we say the design is *balanced*. In this paper, only 2–designs are considered so we will abbreviate 2–(v, k, λ_t) to (v, k, λ).

In a previous paper [4] the author introduced the term *defining set* to refer to a set of blocks which can be a subset of only one t–(v, k, λ_t) design, denoting the defining set by $d(t$–$(v, k, \lambda_t))$.

For example, the set of blocks $R = \{123, 145, 167\}$ can be completed to a (7, 3, 1) design in two distinct ways: by adjoining either $T_1 = \{246, 257, 347, 356\}$ or $T_2 = \{247, 256, 346, 357\}$. Hence R is not a defining set of either design. But the set of blocks $Q = \{123, 145, 246\}$ can be completed to a (7, 3, 1) design only by adjoining the blocks $\{167, 257, 347, 356\}$. Hence Q is a defining set of that design.

A *minimal defining set*, denoted by $d_m(t$–$(v, k, \lambda_t))$, is a defining set no proper subset of which is a defining set. A *smallest defining set*, denoted by $d_s(t$–$(v, k, \lambda_t))$, is a defining set such that no other defining set has smaller cardinality. Clearly, every t–(v, k, λ_t) design has a defining set (the whole design) and hence a smallest defining set. A $d(t$–$(v, k, \lambda_t))$ defining set consisting of blocks of a particular t–(v, k, λ) design D is abbreviated to dD.

The term *trade* is used to refer to two distinct collections of the same number of k–sets which contain precisely the same pairs (see Billington [**1**] and Gray [**3**]); for example, the collections T_1 and T_2 given above. Such collections are also known as *mutually balanced* (Rodger [5]).

Every permutation on the elements of V induces a mapping from a k–set to a k–set. An *automorphism* of a set of blocks X is a permutation on the elements which takes every block of X to a block of X. Let $\mathrm{Aut}(X)$ denote the group of all the automorphisms of X.

In Gray [4], the following results were established for incomplete block designs; they are now given without proof.

LEMMA 1.1. *Every defining set of a t–(v,k,λ_t) design D contains a block of every possible trade $T \subset D$.*

LEMMA 1.2. *Suppose S is a particular defining set of a (v,k,λ) design D and $\rho \in \mathrm{Aut}(D)$. Then $\rho(S)$ is also a defining set of D and $\mathrm{Aut}(S)$ is a subgroup of $\mathrm{Aut}(D)$.*

LEMMA 1.3. *No automorphism of a 2–$(v,k,1)$ design, with $k > 2$, consists of a single transposition.*

LEMMA 1.4. *Any $d(2$–$(v,k,1))$ defining set S, for $k > 2$, has at least $(v-1)$ elements occurring in its blocks.*

LEMMA 1.5. *Suppose each of the elements i and j appears only once in a $d(2$–$(v,k,1))$ defining set S, where $k > 2$. Then i and j cannot appear in the same block of S.*

THEOREM 1.6. *For every 2–$(v,k,1)$ design D, with $k > 2$,*

$$|dD| \geq \frac{2(v-1)}{k+1}.$$

Note that in the case of the $(7,3,1)$ design the bound gives $|dD| \geq \frac{2\times 6}{4} = 3$, and thus the set of three blocks Q given previously must be a smallest defining set.

In considering the effect on the bound of changing the value of λ, it is worth observing that a defining set may have cardinality zero. This is true, for example, of a $d_s(4,3,2)$.

2. Smallest Defining Sets of Affine Planes

An $(n^2,n,1)$ design is also known as an *affine plane of order* n. Affine planes are known to exist whenever n is a power of a prime. The question of whether affine planes exist for other values of n is, in general, open, but the non-existence of an infinite family of affine planes follows from the Bruck-Ryser-Chowla theorem [7]. For the non-existence of the affine plane of order six, the smallest case, also see [6]. Affine planes of orders two, three, four and five are unique up to isomorphism [2].

A *resolution class* of a design is a set of blocks in which each of the v elements appears in exactly one block. A design is said to be *resolvable* if the set of blocks of the design can be partitioned into resolution classes.

The following results are well known for affine planes:

- the (n^2+n) blocks are uniquely resolvable into $(n+1)$ resolution classes, each of n blocks;
- any two blocks from distinct resolution classes intersect in precisely one element.

An affine plane P_1 of order two, or $(4,2,1)$ design, has six blocks in three resolution classes, as follows:

$$12, 34;\ 13, 24;\ 14, 23.$$

Similarly an affine plane P_2 of order three, or (9,3,1) design, has twelve blocks in four resolution classes, as follows:

$$123, 456, 789;\ 147, 258, 369;\ 159, 267, 348;\ 357, 168, 249.$$

Clearly, $|d_s(4,2,1)| = 0$. We now establish a bound for the number of blocks in a defining set of an affine plane in the general case.

LEMMA 2.1. *If S is a defining set of an affine plane of order $n > 2$, and if*

$$|S| = \frac{2(v-1)}{k+1} = 2(n-1),$$

then S consists of $(n-1)$ blocks from each of two resolution classes.

PROOF: Let S be a defining set of an affine plane D of order $n > 2$. The lower bound from Theorem 1.6 has value

$$\frac{2(v-1)}{k+1} = \frac{2(n^2-1)}{(n+1)} = 2(n-1).$$

Suppose $|S| = 2(n-1)$. By Lemmas 1.4 and 1.5, the $2(n-1)n$ entries in the blocks of S must contain at least (n^2-1) elements, with at most $2(n-1)$ elements appearing precisely once. Then there are at least $n^2-2n+1 = (n-1)^2$ remaining elements which must occur at least twice in the remaining $2(n-1)^2$ entries. This is only possible if precisely $(n-1)^2$ elements occur exactly twice.

We now show that the only such subsets of $2(n-1)$ blocks of an affine plane are those formed by taking $(n-1)$ blocks from each of two resolution classes.

Let $\{R_i \mid i = 1,2,\ldots,n+1\}$ be the unique collection of resolution classes of D and let a_i be the number of blocks of S belonging to the ith resolution class, for $i = 1,2,\ldots,n+1$. Now for each a_i,

$$0 \le a_i \le (n-1), \tag{1}$$

since if $a_i = n$ for some i, then all n^2 elements would appear in S. This is impossible since S contains only $2(n-1)+(n-1)^2 = n^2-1$ elements. Clearly,

$$\sum_{i=1}^{n+1} a_i = 2(n-1). \tag{2}$$

Also, let I be the number of pairs of blocks of D intersecting in precisely one element.

$$\begin{aligned} I &= \sum_{i=1}^{n} \left[\sum_{j=i+1}^{n+1} a_i a_j \right] \\ &= \frac{1}{2} \left[\left(\sum_{i=1}^{n+1} a_i \right)^2 - \sum_{i=1}^{n+1} a_i^2 \right] \\ &= \frac{1}{2} \left[4(n-1)^2 - \sum_{i=1}^{n+1} a_i^2 \right] \qquad \text{from (2).} \end{aligned}$$

The minimum possible value of I will occur when $\sum_{i=1}^{n+1} a_i^2$ is a maximum. We find this maximum.

Suppose a set of values $A = \{a_i \mid i = 1, 2, \ldots, n+1\}$ satisfies (1) and (2). Further suppose $1 \leq a_w \leq a_v < (n-1)$ for two particular values v and w. Then the set of values A' formed by replacing a_v by $(a_v + 1)$, a_w by $(a_w - 1)$, and leaving the other values unchanged, must also satisfy (1) and (2).

Now

$$\begin{aligned} (a_v + 1)^2 + (a_w - 1)^2 &= a_v^2 + a_w^2 + 2(a_v - a_w) + 2 \\ &> a_v^2 + a_w^2 \quad \text{since } a_v \geq a_w. \end{aligned}$$

Hence for any such set A with two non-zero values of a_i not equal to $(n-1)$, there exists another set satisfying the conditions with a higher sum of squares. Since a maximum sum of squares exists, it must occur when precisely two resolution classes each contribute $(n-1)$ blocks to S, and the remainder have no blocks in S.

Then I has a minimum value of

$$\frac{1}{2}[4(n-1)^2 - 2(n-1)^2] = (n-1)^2.$$

Only in this case is the minimum number of $(n-1)^2$ pairs of blocks intersecting in one element achieved. Each pair of blocks which intersects in exactly one element corresponds to an element occurring twice in the blocks of S, establishing the lemma. □

THEOREM 2.2. *Every smallest defining set of a* $(9, 3, 1)$ *design* D *contains precisely two blocks from each of two resolution classes of* D.

PROOF: By Lemma 2.1 it is sufficient to show that four blocks of a (9, 3, 1) design, comprising any two blocks of any two resolution classes, form a defining set. Consider the (9, 3, 1) design P_2 given earlier. Any two blocks of a resolution class determine the entire class, and any pair of resolution classes is isomorphic to any other pair. So without loss of generality we take $S = \{123, 456, 147, 258\}$ and suppose that S is a subset of a (9, 3, 1) design D. Completing resolution classes, D must also contain blocks 789 and 369. Pair 15 must occur in a block 159, since pairs 12, 13, 45, 56, 17 and 58 have already occurred. We proceed similarly to the unique design P_2, and hence S is a smallest defining set. □

While the bound given in Theorem 1.6 is achieved for the affine plane of order three, we now show it is never met for affine planes of larger order .

THEOREM 2.3.

$$\text{For } n > 3, \text{ we have } \left|d_s(n^2, n, 1)\right| > 2(n-1).$$

PROOF: Suppose we have a defining set S of an affine plane P of order n on elements $V = \{x_{ij} \mid i, j = 1, ..., n\}$.

By Theorem 1.6, $|S| \geq 2(n-1)$. Suppose $|S| = 2(n-1)$. By Lemma 2.1, S must consist of $(n-1)$ blocks from each of two resolution classes. Since S fully defines these two classes and since any two resolution classes of an affine plane of order n are isomorphic, we can assume without loss of generality that the two classes containing the blocks of S are R_1 and R_2 where the ith element of the jth block of R_1 is x_{ji} and of R_2 is x_{ij}. Suppose R_1 and R_2 belong to some affine plane P.

Consider the permutation

$$\rho = (x_{11}x_{21})(x_{12}x_{22})...(x_{1j}x_{2j})...(x_{1n}x_{2n}), \quad \text{for } j = 1, 2, ..., n.$$

The effect of ρ on R_1 is to interchange the first two blocks, while R_2 undergoes only a reordering of the first two elements in each of its blocks . Hence ρ is an automorphism of $R_1 \cup R_2$

Now P must contain a block b containing pair $x_{41}x_{32}$, since $n > 3$, with $\text{b} \notin R_1 \cup R_2$. The same pair must also belong to $\rho(\text{b})$. Hence if $P = \rho(P)$ it would be necessary that $\text{b} = \rho(\text{b})$ and we show this is impossible. By the intersection property of an affine plane, b must contain elements x_{1l} and x_{2k}, for some unique values l and k, from the first two blocks of R_1 respectively. These elements x_{1l} and x_{2k} are the elements affected by ρ, and thus for $\text{b} = \rho(\text{b})$ we require $\{x_{1l}, x_{2k}\} = \{x_{2l}, x_{1k}\}$ which implies $k = l$. Then pair $x_{1l}x_{2l}$ occurs in $\text{b} \notin R_1 \cup R_2$ and in the lth block of R_2, which is impossible. Hence $\text{b} \neq \rho(\text{b})$ and thus $P \neq \rho(P)$. (It can be similarly shown that all blocks of $P \backslash (R_1 \cup R_2)$ do not belong to $\rho(P)$). Hence $R_1 \cup R_2 \subset P$ and $R_1 \cup R_2 \subset \rho(P)$, giving that S belongs to at least two affine planes of order n. Hence S is not a defining set. Since no defining set S of cardinality $2(n-1)$ can exist, the theorem is proven. □

EXAMPLE 2.4.

$$\text{For } n = 4, \text{ we have } |d_s(16, 4, 1)| = 7.$$

PROOF: By Theorem 2.3, $|d_s(16, 4, 1)| > 6$. We show that a defining set of an affine plane of order four can be formed by taking three blocks from each of two resolution classes and any block from a third class.

Without loss of generality we take the six blocks from the two resolution classes

$$R_1 = \{1234,\ 5678,\ 9ABC,\ DEFG\} \text{ and } R_2 = \{159D, 26AE, 37BF, 48CG\}.$$

We attempt to complete these to a (16, 4, 1) design. Consider element F. Since pairs 12 and $2A$ have occurred two cases are possible.

Case (i).

The remaining blocks containing F are of the form

$$\begin{array}{ll} F1** & \text{with two elements from } \{6, 8, C\}, \\ F2** & \text{with two elements from } \{5, 8, 9, C\}, \\ FA** & \text{with two elements from } \{4, 5, 8\}. \end{array}$$

Element 4 must occur, and cannot occur with 8. Hence we have block $FA45$. Since pairs 86 and $8C$ have occurred we must also have blocks $F16C$ and $F289$. Now consider the two remaining blocks containing 1. These must contain elements $\{7, 8, A, B, E, G\}$. Since pairs 87, $8G$, AE, AB, EG and $7B$ have occurred, the blocks must be $17AG$ and $18BE$. A has now occurred four times, so its final occurrence must be in block $38AD$. Similar considerations for element 3 force blocks $35CE$ and $369G$. We now have three blocks from each of two further resolution classes, forcing blocks $46BD$ and $27CD$, and completion is trivial.

Case (ii). The remaining blocks containing F are of the form

$$F1A*,\ F2** \text{ and } F***,$$

with remaining elements chosen from $\{4, 5, 6, 8, 9, C\}$.

Now element 8 has already occurred with 4, 5, 6 and C. If we have block $F289$, then block $F1A*$ cannot be completed since pairs 14, 15, $A6$ and AC have already occurred. Hence we must have block $F1A8$. Since 2 has occurred with 4 and 6, and C has occurred with 4 and 9, we must have blocks $F25C$ and $F469$.

Now consider the remaining blocks containing 1. Since pairs $6E$, $B7$, BC and EG have occurred, these must be $17CE$ and $16BG$. Element 6 has its final occurrence in block $36CD$ and the remaining blocks are now easily found. Note that this design is obtainable from that given in case (i) by the permutation $\rho = (15)(26)(37)(48)$, and the designs have intersection $R_1 \cup R_2$.

Since two ways of completing the two resolution classes are possible, and blocks of these two classes are the only blocks common to both designs, the proof is complete. □

3. Smallest Defining Sets of Projective Planes

An $(n^2+n+1, n+1, 1)$ design is also known as a *projective plane of order n*. Projective planes are known to have the following properties (see [**2**]):

- any two of the (n^2+n+1) blocks intersect in precisely one element;
- any affine plane of order n can be extended to a projective plane of order n;
- *the residual design*, formed by taking any block and deleting each of its elements from the remaining blocks, is an affine plane of order n.

The $(7,3,1)$ design, given in Section 1, is a projective plane of order two. Projective planes of orders two, three, and four are known to be unique up to isomorphism (see [**2**], p144).

We use the results of Section 2 for affine planes to consider the smallest defining sets of projective planes.

THEOREM 3.1. *If S is a defining set of a projective plane of order $n > 2$, then*

$$|S| \geq 2n > \frac{2(v-1)}{k+1} + 1.$$

PROOF: We consider the cases of $n = 3$ and $n > 3$ separately.

Case (i).

If $n = 3$, suppose S is a $d(13,4,1)$ defining set. Then we need to show

$$|S| \geq 6 > \frac{2 \times 12}{5} + 1.$$

Suppose S consists of five blocks. From Lemmas 1.4 and 1.5, at least twelve of the thirteen elements must appear in S, with no more than five elements occurring exactly once. Let (a_1, a_2, a_3) represent the number of elements occurring once, twice or three times respectively, since it is trivial that no element can appear four times. The only possibilities are $(5,6,1)$ and $(4,8,0)$. The five blocks of S give rise to $\binom{5}{2} = 10$ pairs of blocks, each pair intersecting in exactly one element. Case $(5,6,1)$ accounts for only nine such pairs and case $(4,8,0)$ for only eight. Hence at least $2n = 6$ blocks are necessary.

Case (ii).

Suppose S is a $d(n^2{+}n{+}1, n{+}1, 1)$ defining set, where $n > 3$. Then, by Theorem 1.6,

$$|S| \geq \frac{2(v-1)}{k+1} = \frac{2(n^2+n)}{n+2} = 2(n-1) + \frac{4}{n+2}.$$

Since $n > 3$, this gives $|S| \geq 2n{-}1$.

Now suppose $|S| = 2n{-}1$ and S has blocks b_i, for $i = 1,2,\ldots,2n{-}1$. Suppose b_{2n-1} has elements $e_1, e_2, \ldots, e_{n+1}$. Let S' be the set of blocks obtained by deleting these elements from each b_i, for $i = 1,2,\ldots,2n{-}2$. Now S' consists of $(2n{-}2)$ blocks of an affine plane of order n and, by Theorem 2.3, S' is not a defining set. Hence S' belongs to at least two affine planes, A_1 and A_2 say. Further, two blocks of S' are in the same resolution class of A_1 if and only if they are in the same resolution class of A_2.

Let R^i_j be the jth resolution class of A_i, for $j = 1,2,\ldots,n{+}1$ and $i = 1,2$, where we ensure that R^1_j corresponds to the resolution class obtained by deleting element e_j, and that the blocks of S' in R^1_j are also in R^2_j.

Now take $R_j^i \cup \{e_j\}$ to mean the set of blocks formed by adjoining element e_j to each block of R_j^i, for $j = 1, 2, ..., n+1$, $i = 1, 2$.

Then $S \subset \{R_j^i \cup \{e_j\} \mid j = 1, 2, ..., n+1\} \cup \{b_{2n-1}\}$, for $i = 1, 2$, where each is a distinct projective plane of order n. Thus S is not a defining set and hence no such set of cardinality $2n-1$ exists.

Together, cases (i) and (ii) give the required result. □

EXAMPLE 3.2. *A smallest defining set of a projective plane of order three contains six blocks.*

PROOF: By Theorem 3.1 we need only find a $d_S(13, 4, 1)$ defining set comprising six blocks.

Take the set of blocks

$$S = \{0139,\ 124A,\ 235B,\ 346C,\ 4570,\ 5681\}$$

on elements $\{0, 1, 2, ..., 9, A, B, C\}$.

Elements 1, 3, 4 and 5 have each appeared three times, forcing blocks $378A$, $489B$, $59AC$ and $17BC$. Elements A, C and 9 have now also appeared three times, giving the remaining three blocks 2679, $06AB$ and $028C$ of the $(13, 4, 1)$ design obtained by cycling 0139 modulo 13. □

EXAMPLE 3.3. *A smallest defining set of a projective plane of order four contains eight blocks.*

PROOF: By Theorem 3.1 we need only find a $d_S(21, 5, 1)$ defining set containing eight blocks. Take the set of blocks

0569J	78B02	HIOAC
125FH	67AK1	89C13
9AD24	CDG57	

on elements {0,1,2,...,9,A,B,...,K}.

Consider the remaining blocks containing 0, which must be based on elements {1,3,4,D,E,F,G,K}. Since pairs 4D, DG, 13, 1F and 1K have occurred, these must be 0DF3K and 04GE1.

Since 1 has now occurred in four blocks, its remaining occurrence must be in 1BDIJ. We now have eleven blocks, including all those containing 0 or 1.

Now we consider the residual affine plane formed by deleting the known block 04EG1. This plane has blocks below, arranged by the resolution class resulting from the given deleted element.

0 :	569J	278B	ACHI	DF3K
1 :	25FH	67AK	389C	BDIJ
4 :	9AD2			
G :	CD57			
E :				

Now from the proof of Example 2.4 we know that two resolution classes, together with any additional block, defines a (16,4,1) design. Hence the remaining blocks in the resolution classes above are known. It only remains to adjoin elements `0,1,4,G` and `E` to appropriate classes. We know the classes to be adjoined to `0,1,4` and `G`, and `E` must adjoin the remaining class. The projective plane can thus be uniquely completed, the full set of blocks being those obtained by cycling `014EG` modulo 21. □

4. Smallest Defining Sets of the (6, 3, 2) and (11, 5, 2) Designs

Only designs with $\lambda = 1$, for which Theorem 1.6 applies, have so far been considered in this paper. Smallest defining sets of 2–(7,3,2) designs were dealt with in [4]. Results are now given for two further designs with $\lambda = 2$.

EXAMPLE 4.1. *A smallest defining set of a* $(6,3,2)$ *design contains three blocks.*

PROOF: Consider the three blocks 02∞, 13∞ and 03∞ in a design based on elements $\{0,1,2,3,4,\infty\}$. If these belong to a $(6,3,2)$ design, then so must blocks 14∞ and 24∞. Now consider the remaining blocks containing 4; these must be 043, $04*$ and $4**$ with remaining elements chosen from $\{1,2,3\}$. Pair 03 has already occurred twice, and if we take block 042 the remaining block containing 0 must contain element 1 twice. Hence we must have 043, 041 and 423, leading to final blocks 012 and 123. This is the design obtained by cycling blocks 012 and 02∞ modulo 5.

To show that no smaller defining set exists, consider each possible pair of blocks of the design above. For any such pair S we find an automorphism of S which is not an automorphism of the full design; for example, (03) is an automorphism of blocks $S = \{012, 123\}$ which takes block 02∞ to 23∞, not in the design. By Lemma 1.2, S is not a defining set.

Such a permutation can be found for all pairs of blocks and hence at least three blocks are required for a defining set. Since the $(6,3,2)$ design is unique up to isomorphism (see, for instance, [**7**]) the proof is complete. □

We now consider the $(11,5,2)$ design, which is also unique up to isomorphism. We examine the particular design D obtained by cycling 13459 modulo 11 on elements $\{0,1,2,...,9,A\}$.

LEMMA 4.2. *A defining set of an* $(11,5,2)$ *design contains at least five blocks.*

PROOF: Suppose there is a defining set consisting of four blocks. Two cases are possible.

Case (i). There is an element common to all four blocks.

Without loss of generality, let the element 1 occur in each of the four blocks of a defining set. It is immaterial which four blocks are chosen, as in any event we know all five blocks containing 1, namely 13459, 46781, $8A015$, $A1237$ and 90126. Then the remaining six blocks can be found in only two ways:

$$T_3 = \{2456A, 24830, 25879, 4079A, 869A3, 50367\};$$
$$T_4 = \{24570, 248A9, 25836, 46A03, 80379, 56A79\}.$$

Hence such a set of four blocks cannot be a defining set.

Case (ii). No element is common to all four blocks.

We show that some element occurs three times. Without loss of generality, take the first two blocks to be 13459 and 2456A, since any other two blocks can be taken to these under the automorphism

$$f(x) = ax + b, \text{ for } a = 3, 9, 5, 4, 1 \text{ and } b \in \mathbb{Z}_{11}.$$

If 4 or 5 are not to occur three times, the remaining two blocks must be chosen from 689A3, 90126 and A1237, resulting in each case in some element occurring three times. We can assume that element 1 occurs in precisely three of the four blocks, including 13459 and 46781. Then for each other choice of a block containing 1 and a block not containing 1 we find an automorphism of the four blocks which is not an automorphism of the full design D; for example, blocks 13459, 46781, 8A015 and 2456A have automorphism $(39) \notin \mathrm{Aut}(D)$.

It can be verified that every such set of four blocks has an automorphism which is a transposition, and thus cannot be a defining set of this design D. Cases (i) and (ii) give the lemma. □

EXAMPLE 4.3. *A smallest defining set of an* (11, 5, 2) *design contains five blocks.*

PROOF: We refer to case (i) of Lemma 4.2. An (11, 5, 2) defining set can be obtained by taking any four blocks containing 1, together with a single block of T_3; for example, 13459, 46781, 8A015, A1237 and 2456A. Together with Lemma 4.2, the proof is complete. □

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