THE CENTRE OF A SLOOP

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ABSTRACT

Every Steiner triple systems can be co-ordinatised in terms of a commutative loop, called a sloop. Let $(T, \circ)$ be such a sloop. The set of all elements $a \in T$ such that, for all $x, y \in T$, $a\circ(x\circ y) = (a\circ x)\circ y$ is called the centre of T. We prove that a given element a belongs to the centre of T if and only if, for each pair of distinct elements $x, y \in T\backslash\{e, a\}$, the three elements a, x, y are contained in at least one subsystem of order 7. Further we give an alternate set of necessary conditions for the existence of a non-trival centre; namely $v \equiv 3, 7(mod\ 12)$.

Let S be a set of size v. Then a *Steiner triple system*, $\mathcal{S}$, is a collections of 3-subsets chosen from S in such a way that each pair of distinct elements $x, y \in S$ occurs in precisely one 3-subset. The 3-subsets are called the blocks of $\mathcal{S}$. The order of the Steiner triple system is v, the size of the set S. It has been shown that necessary and sufficient conditions for the existence of a Steiner triple system is that $v \equiv 1$ or $3(mod\, 6)$. Simple counting arguments show that; each element must occur in $\dfrac{v-1}{2}$ blocks, and the Steiner triple system is made up of $\dfrac{v(v-1)}{6}$ blocks. A Steiner triple system $\mathcal{R}$ is said to be a *subsystem* of $\mathcal{S}$ if, $R \subseteq S$ and the blocks of $\mathcal{R}$ are also blocks of $\mathcal{S}$. In other words $\mathcal{R} \subseteq \mathcal{S}$.

If we take the underlying set S of a Steiner triple system, then we can define a binary operation based on the incidence of the elements within the blocks of the design. The binary operation can be defined in such a way that the corresponding algebra forms a commutative idempotent quasigroup called a squag. These systems are discussed in [**2**] pp.3-4. However if we adjoin a neutral element to S, every Steiner triple system can be co-ordinatised in terms of a commutative loop, called a sloop. It is know that a Steiner triple system on 7 points is a projective plane. The corresponding sloop forms a group, (see [**2**] pp.19), and hence is associative. Not all sloops are associative and so we define the centre of a sloop to be the set of elements which associate with elements x, y, for all x, y of the sloop. We show that a sloop contains a non-trival centre if its corresponding Steiner triple system has an appropriate number of subsystems of order 7. Thus we give necessary and sufficient conditions for the existence of a non-trival centre.

First we give a formal definition of a sloop and its centre.

DEFINITION 1,[**2**], pp.3-4: Let $\mathcal{S}$ be a Steiner triple system defined on the set S. Let $T = S \cup \{e\}$, where $e \notin S$. Define a binary operation $\circ$ on T by:

(1) if $x, y \in S$ and $x \neq y$, then $x \circ y = z$, where $\{x, y, z\} \in \mathcal{S}$;

(2) for all $x \in T$, $x \circ e = x = e \circ x$;

(3) for all $x \in T$, $x \circ x = e$.

Then $(T, \circ)$ is a commutative loop, known as a *Steiner loop* or *sloop*.

The commutativity of $\circ$ follows naturally. Also it is obvious that whenever $x \circ y = z$, then $x \circ z = y$ and $y \circ z = x$. Consequently $x \circ (x \circ y) = y$, for all $x, y \in S$. These identities form a basis for the variety of all sloops.

THEOREM 2,[**2**], pp.3-4. *A basis for the variety of all sloops is given by:*

(1) $x \circ e = x$,

(2) $x \circ y = y \circ x$,

(3) $x \circ (x \circ y) = y$,

where $x, y \in S$.

Bruck [**1**], pp.57 defines the nucleus of a loop to be all elements a of the loop such that, for all elements x, y of the loop, $(a \circ x) \circ y = a \circ (x \circ y)$, $(x \circ a) \circ y = x \circ (a \circ y)$ and $(x \circ y) \circ a = x \circ (y \circ a)$. He goes on to define the centre of the loop to be all elements a of the nucleus such that $a \circ x = x \circ a$, for all x in the loop. A Steiner loop or sloop is commutative on all elements. Hence we give the following definition for the center of a sloop.

DEFINITION 3: Let T be any sloop. The set of all $a \in T$ such that, for all $x, y \in T$,

$$a \circ (x \circ y) = (a \circ x) \circ y$$

is defined to be the *centre* of T.

The following theorem gives necessary and sufficient conditions for the existence on a non-trival centre of a sloop.

THEOREM 4. *Let $(T, \circ)$ be a sloop based on a Steiner triple system $\mathcal{S}$, with $|S| \geq 7$. A given element a belongs to the centre of T if and only if for each pair of distinct elements $x, y \in T \setminus \{e, a\}$, the three elements a, x, y are contained in at least one subsystem, of order 7, of $\mathcal{S}$.*

PROOF: Assume there exists an element $a \in T$ such that, for all $x, y \in T$, $a \circ (x \circ y) = (a \circ x) \circ y$. Choose any pair of distinct elements $x, y \in T \setminus \{e, a\}$. Then there are two cases to consider. Case 1): The three elements a, x, y occur together as a block of the corresponding Steiner triple system, $\mathcal{S}$. Case 2): The three elements a, x, y do not constitute a block of $\mathcal{S}$.

Case 1): Recall $|S| \geq 7$. If the Steiner triple system contains the block $a\,x\,y$, then there exist elements $r, s \in S$ such that $\mathcal{S}$ must also contain blocks of the form:

$a\,x\,y$	$a\,r\,s$
$x\,r\,\alpha$	$x\,s\,\beta$
$y\,r\,\gamma$	$y\,s\,\delta$
$a\,\alpha\,\rho$	$a\,\beta\,\sigma$,

for suitable elements α, β, γ, δ, ρ, σ of S. It follows from the properties of a Steiner triple system that element α and β must be distinct. However we go on to show that $\sigma = \delta = \alpha$ and $\rho = \gamma = \beta$. At this point the block $a\,\alpha\,\rho$ can be rewritten as $a\,\alpha\,\beta$ or as $a\,\sigma\,\beta$. A Steiner triple system contains no repeated blocks. Hence the blocks $a\,\alpha\,\rho$ and $a\,\beta\,\sigma$ are in fact the same block and it occurs precisely once.

Let us begin the proof by considering the three elements a, x, α. We have assumed that

$$(a\circ x)\circ\alpha = a\circ(x\circ\alpha).$$

But $a\circ x = y$ and $x\circ\alpha = r$ so,

$$y\circ\alpha = a\circ r.$$

We also know that $a\circ r = s$, implying

$$y\circ\alpha = s \quad \text{or} \quad y\circ s = \alpha.$$

However in the above blocks we have set $y\circ s = \delta$, so $\delta = \alpha$. Now take the three elements a, x, β. Applying a similar procedure we obtain

$$(a\circ x)\circ\beta = a\circ(x\circ\beta)$$
$$y\circ\beta = a\circ s = r \quad \text{or} \quad y\circ r = \beta.$$

In the original blocks we set $y\circ r = \gamma$, and so $\gamma = \beta$. Finally take the three elements a, r, α. We observe that

$$(a\circ\alpha)\circ r = a\circ(\alpha\circ r)$$
$$\rho\circ r = a\circ x = y \quad \text{or} \quad r\circ y = \rho.$$

Once again the above blocks give $r\circ y = \gamma = \beta$ and so $\rho = \beta$ and $\sigma = \alpha$. This gives the distinct blocks

$a\,x\,y$	$a\,r\,s$
$x\,r\,\alpha$	$x\,s\,\beta$
$y\,r\,\beta$	$y\,s\,\alpha$
$a\,\alpha\,\beta$.	

This is a subsystem, of order 7, based on the set $\{a, x, y, r, s, \alpha, \beta\}$.

Case 2): Assume the three elements a, x, y do not constitute a block of S. Since $|S| \geq 7$, there exists suitable elements r, s, t, α, β, γ of S such that S contains blocks of the form:

$a\,x\,r$	$a\,y\,s$
$x\,y\,t$	$x\,s\,\alpha$
$y\,r\,\beta$	$r\,s\,\gamma$,

where the elements x, y, r, s, t, α are all distinct. We will show that $\beta = \alpha$ and $\gamma = t$. Further we show that these blocks imply the existence of a block of the form $a\,t\,\alpha$.

Consider the three elements a, x, α. We have assumed that

$$(a \circ x) \circ \alpha = a \circ (x \circ \alpha)$$
$$r \circ \alpha = a \circ s = y \quad \text{or} \quad y \circ r = \alpha.$$

But $y \circ r = \beta$, therefore $\beta = \alpha$. Now take the three elements a, x, y. We obtain

$$a \circ (x \circ y) = (a \circ x) \circ y$$
$$a \circ t = r \circ y = \beta = \alpha.$$

It follows that S must contain the block $a\,t\,\alpha$. Finally we consider the three elements a, x, s and equations

$$a \circ (x \circ s) = (a \circ x) \circ s$$
$$a \circ \alpha = r \circ s = \gamma.$$

However the inclusion of the block $a\,t\,\alpha$ implies that $\gamma = t$. It follows that S must contain the blocks

$a\,x\,r$	$a\,y\,s$
$x\,y\,t$	$x\,s\,\alpha$
$y\,r\,\alpha$	$r\,s\,t$
$a\,\alpha\,t.$	

These seven blocks from a subsystem, based on the seven elements a, x, y, r, s, t, α.

Conversely, assume that for any pair of distinct elements $x, y \in T \setminus \{e, a\}$, the three elements a, x, y are contained in at least one subsystem of S, where the order of the subsystem is 7. Fix such a subsystem and denote it by S_0. Up to isomorphism there is only one Steiner triple system on seven points and this is a projective plane. As stated earlier, the corresponding sloop is associative. Hence, for all $r, s, t \in S_0$, $r \circ (s \circ t) = (r \circ s) \circ t$. It is immediate from this that

$$(a \circ x) \circ y = a \circ (x \circ y).$$

We note that if $x = y$ or $x = a$ or e, then, by the properties of a sloop, $(a \circ x) \circ y = a \circ (x \circ y)$. Hence the element a is a member of the centre of the sloop.

The result now follows.

The above theorem can be restated in terms of loops and subgroups of order 8. It should be noted that this subgroup is the elementary abelian group on 8 elements.

THEOREM 5. *Let $(T, \circ)$ be a totally symmetric loop. A given element a belongs to the centre of T if and only if for each pair of distinct elements $x, y \in T \setminus \{e, a\}$, the three elements a, x, y are contained in at least one subgroup, of order 8, of T.*

We illustrate Theorem 4 with the following examples.

EXAMPLE 6: The projective spaces over $GF[2]$ corresponds to Steiner triple systems, of order $2^n - 1$. It is relatively easy to show that any three points of a projective space, over $GF[2]$, generate a projective plane of order 7. The subvariety of sloops defined by the projective space, over $GF[2]$, is characterized, among all sloops, by the associative law, (see [**2**] pp.19). For each of these sloops the centre is the entire set S.

EXAMPLE 7: [**3**] pp.19 Take the Steiner triple system on 15 points with the following blocks:

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
2 4 6	2 5 7	2 8 10	2 9 11	2 12 14	2 13 15	3 4 7
3 5 6	3 8 11	3 9 10	3 12 15	3 13 14	4 8 12	4 9 13
4 10 14	4 11 15	5 8 13	5 9 12	5 10 15	5 11 14	6 8 15
6 9 14	6 10 13	6 11 12	7 8 14	7 9 15	7 10 12	7 11 13.

This Steiner triple system contains exactly 7 subsystems, of order 7, each of which is listed below.

R= $\{1, 2, 3, 4, 5, 6, 7\}$

Blocks			
	1 2 3	1 4 5	1 6 7
	2 4 6	2 5 7	3 5 6
	3 4 7		

R= $\{1, 2, 3, 8, 9, 10, 11\}$

Blocks			
	1 2 3	1 8 9	1 10 11
	2 8 10	2 9 11	3 9 10
	3 8 11		

R= $\{1, 2, 3, 12, 13, 14, 15\}$

Blocks			
	1 2 3	1 12 13	1 14 15
	2 12 14	2 13 15	3 13 14
	3 12 15		

R= $\{1, 4, 5, 8, 9, 12, 13\}$

Blocks			
	1 4 5	1 8 9	1 12 13
	4 8 12	4 9 13	5 9 12
	5 8 13		

R= $\{1, 4, 5, 10, 11, 14, 15\}$

Blocks			
	1 4 5	1 10 11	1 14 15
	4 10 14	4 11 15	5 11 14
	5 10 15		

R= $\{1,6,7,8,9,14,15\}$

Blocks	1 6 7	1 8 9	1 14 15
	6 8 15	6 9 14	7 9 15
	7 8 14		

R= $\{1,6,7,10,11,12,13\}$

Blocks	1 6 7	1 10 11	1 12 13
	6 10 13	6 12 11	7 10 13
	7 11 13		

It is easy to check that the element 1 is a member of each of the above subsystems and that each of the three element sets $1, x, y$, where $x \neq y$ and $x, y \in \{2, \ldots, 15\}$, occurs in at least one of the subsytems above. A little bit of work will show that for all $x, y \in \{1, \ldots, 15\} \cup \{e\}$, $1\circ(x\circ y) = (1\circ x)\circ y$. (This fact was originally found by a computer program.) If we consider any of the elements $2, \ldots, 15$, then these elements do not associate with all other elements of the sloop. For example $2\circ(8\circ 12) = 2\circ 4 = 6$, while $(2\circ 8)\circ 12 = 10\circ 12 = 7$. Hence these elements do not belong to the center of the sloop. Therefore the corresponding sloop has a non-trivial centre consisting of the elements $e, 1$.

Assume we have a Steiner triple system which contains two subsystems, of order 7, both of which contain the blocks $a\,b\,c$ and $a\,d\,e$. The original Steiner triple system must contain blocks of the form, $b\,d\,.$, $b\,e\,.$, $c\,d\,.$ and $c\,e\,.$. In addition these blocks must occur in both subsystems. It now follows that the two subsystems are in fact the same subsystem. Hence any two distinct subsystems, of order 7, have at most one block in common. Further, it follows that any two subsystems, of order 7, have at most three elements in common.

We will go on to apply a counting argument to the results stated in Theorem 4. Hence we obtain alternate necessary conditions for the existence of a non-trival centre. We show that the order of the Steiner triple system must be congruent to $3, 7 (mod\ 12)$.

Assume we have a Steiner triple system which has a corresponding sloop with a non-trival centre. Further assume element a of the Steiner triple system is a member of the centre. In any Steiner triple system the element a must occur in $\frac{v-1}{2}$ blocks. Represent these blocks as follows:

$$a\,x_1\,y_1$$
$$a\,x_2\,y_2$$
$$\vdots$$
$$a\,x_n\,y_n,$$

where $n = \frac{v-1}{2}$. If the elements a, x_1 occur in a subsystem, of order 7, then such a subsystem must contain the block $a\,x_1\,y_1$ and hence all three elements a, x_1, y_1.

If we consider the three elements a, x_1, z, for all $z \neq a, x_1, y_1$, then by Theorem 4 these three elements must occur together in at least one subsystem, of order 7. As stated above, any two subsytem, of order 7, have at most three elements in common. Each of these subsystems already contains the elements a, x_1, y_1 and so the remaining four elements must be distinct. Therefore there are $v - 3$ possible choices for z to be partitioned into sets of size four. Hence $v \equiv 3(mod\, 4)$, but we already know that $v \equiv 1, 3(mod\ 6)$. Putting these two results together we obtain the following corollary.

COROLLARY 8: Let $(T, \circ)$ be a sloop based on Steiner triple system $\mathcal{S}$, with $|S| \geq 7$. If T contains a non-trival centre, then the order of the Steiner triple system, $\mathcal{S}$, is congruent to $3, 7(mod\ 12)$.

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