On the construction of n-dimensional Designs from 2-dimensional Designs

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ABSTRACT: Let H be an abelian group of order v. If $X = (f(h_1 + h_2))$ $(h_1,h_2 \in H)$ is a $v \times v$ design, then $X = (f(h_1 + h_2 + \ldots + h_n))$ is a proper n-dimensional design. A difficulty with this construction is that it can only be applied to a small number of (2-dimensional) designs. This paper develops a very general technique for generating a proper n-dimensional design from 2-dimensional designs. Indeed, it is shown that Drake's generalised Hadamard matrices, Berman's nega-cyclic and ω-cyclic (generalised) weighing matrices and both of the orthogonal designs of order 4 and type (1,1,1,1) can be extended to give proper n-dimensional designs. In addition, this technique leads to a representation of 2-dimensional designs which generalises the concept of a difference set. This representation is interesting because of its brevity and its wide applicability.

## 1. INTRODUCTION

Let H be a group of finite order v, and let S be a set. Also, let X be a $v \times v$ matrix, indexed over H, of elements of S such that, for some $f: H \rightarrow S$, $X = (f(h_1 + h_2))$ $(h_1,h_2 \in H)$. In this paper, we say X is a *matrix developed modulo the group* H. Now, let $X_n = (f(h_1 + h_2 + \ldots + h_n))$ ($X_n$ is a $v \times v \times \ldots \times v$ n-dimensional array of elements of S), and consider the 2-dimensional subarrays of $X_n$ obtained by fixing all but two indicies. We call these subarrays *sections* of $X_n$. Hammer and Seberry (1979, Theorem 4) noted that, if $X = (f(h_1 + h_2))$ is a Hadamard matrix which can be developed modulo an abelian group, then $X_n = (f(h_1 + h_2 + \ldots + h_n))$ is a proper n-dimensional Hadamard matrix. Indeed, $X_n$ will have the same rows and columns as X. Hence, if X is a design whose defining properties depend only on its set of rows, then $X_n$ is an n-dimensional array whose faces are all designs of the same sort. An n-dimensional design whose sections are designs of a given sort is said to be a *proper n-dimensional design*

This generalisation of Hammer and Seberry's construction is attractive because of its simplicity and because of its applicability to different kinds of designs. However, designs which are developed modulo a group are often not available. Indeed, for certain types of designs, concepts which are slightly different to the development of matrices modulo a group are more useful in generating families of designs. For example, Berman (1978) exhibits large families of negacyclic weighing matrices and ω-cyclic generalised weighing matrices (negacyclic and ω-cyclic matrices are similar to cyclic matrices except that as an entry is shifted to the first column it is multiplied by −1 and ω respectively). Delsarte, Goethals and Seidel (1971) construct negacyclic conference matrices, and many Hadamard matrices are obtained by combining negacyclic and cyclic matrices.

This paper extends Hammer and Seberry's construction to obtain a method of developing proper n-dimensional designs from large classes of many types of 2-dimensional designs. In particular, the technique applies to all the designs mentioned in the previous paragraph as well as to well known orthogonal designs and generalised Hadamard matrices which are not developed modulo a group.

In the next section, we formalise the concept of a design whose defining properties depend only on its set of rows and columns, and we introduce the idea of a *collapsable* function. In section three, we develop the concept of an *extension* function, and show how to construct families of collapsable functions. Finally, in section four, we show that our technique applies to Drake's (1979) generalised Hadamard matrix, to the two orthogonal designs of order four and type (1,1,1,1), and to any negacyclic or ω-cyclic design.

## 2. COLLAPSABLE FUNCTIONS AND DESIGNS

**Definition 2.1:** Let v be an integer, and let S be a finite set. Also, let $\Pi_R$ and $\Pi_C$ be two permutation groups on S. A $(v, \Pi_R, \Pi_C, \beta, S)$ - *design* is a $v \times v$ array, X, of elements of S in which every pair of distinct rows obeys a set of "balance" rules, $\beta$, which are invariant under

- (i) the permutation of a row or column of X,
- (ii) the application of any map in $\Pi_R$ to all of the entries in any row of X, and
- (iii) the application of any map in $\Pi_C$ to all of the entries in any column of X.

When $\Pi_C = \Pi_R = \Pi$, we use the notation "$(v,\Pi,\beta,S)$ - design".

**Example 2.2:** An SBIBD $(v,k,\lambda)$ is a $(v,\Pi,\beta,S)$ - design where

- (i) $S = \{0,1\}$,
- (ii) $\Pi_R = \Pi_C = \{\text{identify map}\}$,
- (iii) $\beta$ is the set of rules: (1) each row and column contains precisely k non-zero entries; and (2) each pair of distinct rows has inner product equal to $\lambda$.

**Example 2.3:** An *orthogonal design* of order v and type $(s_1,s_2,\ldots,s_l)$ $(s_1>0)$ on the commuting variables $x_1,x_2,\ldots, x_l$ is a $(v,\Pi, \beta,S)$ - design where

- (i) $S = \{\pm x_i \mid i = 1,2,\ldots,l\}$,

(ii) $\Pi_R = \Pi_C = \{\pi_1, \pi_{-1}\}$, where $\pi_x$ is multiplication by x,

(iii) $\beta$ is the set of rules: (1) each row and column contains $\pm x_i$ precisely $s_i$ times ($i=1,\ldots,l$); and (2) each pair of distinct rows has inner product equal to zero.

**Example 2.4:** If G is a group of order g, then a *balanced generalised weighing matrix*, BGW(v,k,λ,G) (see de Launey (1984) for a definition), is a $(v,\Pi_R,\Pi_C,\beta,S)$-design where

(i) $S = \{0\} \cup G$

(ii) $\Pi_R = \{\rho_h \mid h \in G\}$ where $\rho_h : S \to S$ is premultiplication by h, and $\Pi_C = \{\sigma_h \mid h \in G\}$ where $\sigma_h : S \to S$ is postmultiplication by h.

(iii) $\beta$ is the set of rules : (1) each row and column contains exactly k non-zero entries; and (2) over Z(G) (the group ring of G over the integers Z), each pair of distinct rows has inner product equal to $\lambda G/g$.

Other examples are generalised Hadamard matrices (GH(n;G)), generalised weighing matrices (GW(n,k;G)) (see de Launey (1984) for definitions), and weighing matrices (W(n,k)).

**Definition 2.5:** Let X and Y be $(v, \Pi_R, \Pi_C, \beta, S)$ - designs. We say X is *equivalent* to Y if X can be obtained from Y by applying a sequence of operations (i), (ii) and (iii) of Definition 2.1.

Often the transpose of a $(v, \Pi_R, \Pi_C, \beta, S)$- design is also a $(v, \Pi_R, \Pi_C, \beta, S)$- design.

**Definition 2.6:** A $(v, \Pi_R, \Pi_C, \beta, S)$ - design. X, is *transposable* if its transpose, $X^T$, is also a $(v, \Pi_R, \Pi_C, \beta, S)$ - design.

Ordinary Hadamard and weighing matrices, orthogonal designs and SBIBD's are all transposable designs. Also, if G is abelian, GH(n;G), BGW(v,k,λ,G) and GW(v,k;G) are transposable. However, if G is non-abelian these designs may not be transposable.

**Definition 2.7:** A $v \times v \times \ldots \times v$ (n times) array, X, of elements of S is said to be a *proper n-dimensional design*, $(v,\Pi_R,\Pi_C,\beta,S)^n$ - design, if, for every 2-dimensional submatrix, A, of X which is obtained by fixing all but two ordinates, either A or $A^T$ is a $(v,\Pi_R,\Pi_C,\beta,S)$ - design. We call these submatrices *sections*.

We now make a definition basic to our construction of proper n-dimensional designs. If S is a set, we let $P_S$ denote the set of invertible maps $f : S \to S$.

**Definition 2.8:** Let H be a finite group; let S be a set, and let $P_S$ denote the set of invertible maps on S. Let $\mathcal{H}$ be the set of all finite tuples of elements of the additive group H. We say $f : \mathcal{H} \rightarrow S$ is *collapsable* if it has the following properties:

(i) For each integer $n \geq 2$, there exists a function $p_n : H \times H \rightarrow P_S$ such that $f(h_1,h_2,\ldots,h_n) = p_n(h_1,h_2)(f(h_1 + h_2,h_3,\ldots,h_n))$;

(ii) For each integer $n \geq 2$, there exists a function $q_n : H \times H \rightarrow P_S$, such that $f(h_1,h_2,\ldots,h_n) = q_n(h_{n-1},h_n)(f(h_1,h_2,\ldots,h_{n-1} + h_n))$.

We shall return to discuss these functions later. For the moment let us consider their utility.

**Theorem 2.9:** *Suppose* $f: \mathcal{H} \rightarrow S$ *is collapsable, and that* $X_2 = (f(h_1, h_2))$ *is a* $(v,\Pi_R,\Pi_C,\beta,S)$-*design. Also, suppose* $p_i(H \times H) \subset \Pi_R$ *and* $q_i(H \times H) \subset \Pi_C$ $(i = 1,\ldots,n)$; *then* $X_n = (f(h_1, h_2,\ldots,h_n))$ *is a proper* $(v,\Pi_R,\Pi_C, \beta, S)^n$ - *design.*

**Proof** It is sufficient to prove that every 2 - dimensional sub matrix of the form

$$Y = (y_{h_i h_j}) = (f(h_1,h_2,\ldots,h_i,\ldots,h_j,\ldots,h_n)) \qquad (j > i)$$

is a $(v,\Pi_R,\Pi_C, \beta, S)$ - design. Now put

$$Q(h_j) = q_{n-j+2}(h_{n-1},h_n) \circ q_{n-j+1}(h_{n-2},h_{n-1} + h_n) \circ\ldots\circ q_3(h_j,h_{j+1} +\ldots+ h_n),$$

$$P(h_i) = p_n(h_1,h_2) \circ p_{n-1}(h_1 + h_2,h_3) \circ\ldots\circ p_{n-j+3}(h_1, + \ldots+ h_{j-2},h_{j-1});$$

then

$$y_{h_i h_j} = P(h_i) \circ Q(h_j)(f(h_1, +\ldots+ h_{j-1},h_j +\ldots+ h_n)).$$

So Y is equivalent to X. This completes the proof.

## 3. EXTENSION FUNCTIONS AND COLLAPSABLE FUNCTIONS

Our aim will be to obtain a widely applicable way of constructing, via *extension functions* (see Definition 3.2), collapsable functions which are derived from suitably structured 2-dimensional designs. So, although any collapsable function, which meets the conditions of Theorem 2.9, implies the existence of a nicely structured n-dimensional design, we will only study a subset of these functions. In particular, the collapsable functions will be *uniform* in the sense that, for some $p: H \times H \rightarrow S$,

$$q_i(h_1, h_2) = p_j(h_1, h_2) = p(h_1, h_2) \quad i,j = 1,2,\ldots$$

Throughout this section, we will use the following notation.

**Notation 3.1:** *Let* H, $\mathcal{H}$, $P_S$ *and* S *be as defined in Definition 2.8. If* p: $H \times H \rightarrow P_S$ *is a map and* f: $H \rightarrow S$ *is a map, we let* $C_{p,f}: \mathcal{H} \rightarrow S$ *and* $D_{p,f}: \mathcal{H} \rightarrow S$ *be defined iteratively as follows : for* $n \geq 2$ *and* $h_1, h_2, \ldots, h_n \in H$, *set*

$$C_{p,f}(h_1) = D_{p,f}(h_1) = f(h_1),$$

*and put*

$$C_{p,f}(h_1,h_2,\ldots,h_n) = p(h_1,h_2)(C_{p,f}(h_1 + h_2,h_3,h_4,\ldots,h_n))$$
$$D_{p,f}(h_1,h_2,\ldots,h_n) = p(h_{n-1},h_n)(D_{p,f}(h_1,h_2,\ldots,h_{n-2},h_{n-1} + h_n)).$$

*Also, we use* $\prod_{i=1}^{n}$ *to denote an indexed composition of maps in the same way that* $\prod_{i=1}^{n}$ *denotes an indexed summation. So*

$$C_{p,f}(h_1,h_2,\ldots,h_n) = \left[\prod_{i=2}^{n} p(\sum_{j=1}^{i-1} h_j,h_i)\right](f(\sum_{j=1}^{n} h_j)), \quad \textit{and}$$

$$D_{p,f}(h_1,h_2,\ldots,h_n) = \left[\prod_{i=n-1}^{1} p(h_i, \sum_{j=i+1}^{n} h_j)\right](f(\sum_{j=1}^{n} h_j)),$$

*The ordering of the indices is important;* $\prod_{i=1}^{n} p_i = p_1 \circ p_2 \circ \cdots \circ p_n$ *may not equal* $\prod_{i=n}^{1} p_i = p_n \circ p_{n-1} \circ \cdots \circ p_1$

**Definition 3.2:** Let p: $H \times H \rightarrow P_S$ be a map. We say p is an *extension function* if, for all f: $H \rightarrow S$, $C_{p,f}$ is collapsable. If p(0,0) is the identity map, then p is said to be a *normalised extension* function.

**Lemma 3.3:** *Let* H *be an additive group,* S *be a set, and* $P_S$ *be the set of invertible maps on* S. *Also, suppose* $p : H \times H \rightarrow P_S$ *is an extension function with* Z = p(0,0). *Then, for all* $h_1,h_2,h_3 \in H$,

$$p(h_1,0) = p(0,h_2) = Z, \tag{3.1}$$
$$Z \circ p(h_2,h_3) = p(h_2,h_3) \circ Z, \tag{3.2}$$
$$p(h_1,h_2) \circ p(h_1 + h_2,h_3) = p(h_2,h_3) \circ p(h_1, h_2 + h_3), \tag{3.3}$$

*and, in particular,* $Z^{-1} \circ p$ *is a normalised extension function.*

**Proof.** Now, for all $f : \mathcal{H} \rightarrow S$, $C_{p,f} : H \rightarrow S$ is collapsable. Hence, by Definitions 3.2 and 2.8,

$$p(h_1,h_2) \circ p(h_1 + h_2,h_3)(f(h_1 + h_2 + h_3)) = p(h_2,h_3) \circ p(h_1,h_2 + h_3)(f(h_1 + h_2 + h_3)).$$

But, since f is arbitrary, this means

$$p(h_1,h_2) \circ p(h_1 + h_2,h_3) = p(h_2,h_3) \circ p(h_1,h_2 + h_3),$$

for all $h_1,h_2,h_3 \in H$. This proves (3.3).

Now put $h_1 = h_2 = 0$ and $h_2 = h_3 = 0$ in (3.3). We obtain

$$Z \circ p(0,h_3) = p(0,h_3) \circ p(0,h_3),$$
$$p(h_1,0) \circ p(h_1,0) = Z \circ p(h_1,0),$$

and, since $p(h_1,h_2)$ is always invertible, we obtain (3.1). Finally, to obtain (3.2), put $h_1 = 0$ in (3.3), and use (3.1).

**Theorem 3.4:** *Let H be an additive group, S be a set, $P_S$ be the set of invertible maps on S, and let $p : H \times H \to P_S$ be a map. The following four statements are equivalent.*

(a) *p is an extension map.*
(b) *For any map $f : H \to S$, $C_{p,f}$ is collapsable.*
(c) *For any map $f : H \to S$, $D_{p,f}$ is collapsable.*
(d) *For any map $f : H \to S$, $C_{p,f} = D_{p,f}$.*

**Proof** (a)$\Leftrightarrow$(b) : By definition.
(b)$\Rightarrow$(d) and (c): For all $f : H \to S$, $C_{p,f}$ is collapsable. So, for all $i \geq 2$, there exist $q_i : H \times H \to P_S$ such that, for all $h_1, h_2,\ldots,h_i \in H$,

$$C_{p,f}(h_1, h_2,\ldots,h_i) = q_i(h_{i-1},h_i)(f(h_1,\ldots,h_{i-2},h_{i-1} + h_i)) \quad (3.4)$$

Hence, for all $h_1, h_2, \in H$,

$$p(h_1,h_2)(f(h_1 + h_2)) = q_2(h_1,h_2)(f(h_1 + h_2)).$$

But f is arbitrary; so $q_2 = p$. Now suppose $q_i = p$, for $i = 2,\ldots,n-1$ then, applying (3.4) iteratively with $i = 2,\ldots, n$, we have

$$p(h_1,h_2) \circ\ldots\circ p(h_1 + \ldots + h_{n-1},h_n) = q_n(h_{n-1},h_n) \circ p(h_{n-2},h_{n-1} + h_n) \circ\ldots\circ p(h_1,h_2 + \ldots + h_n).$$

Now put $h_1 = h_2 = \ldots = h_{n-2} = 0$. By (3.1) and (3.2),

$$p(h_{n-1},h_n) \circ Z^{n-2} = Z^{n-2} \circ p(h_{n-1},h_n) = q_n(h_{n-1},h_n) \circ Z^{n-2}.$$

Hence, because $Z^{n-2}$ is invertible, $q_n = p$. So, by (3.4),

$$\begin{aligned} C_{p,f}(h_1,h_2,\ldots,h_n) &= p(h_{n-1},h_n) \circ p(h_{n-2},h_{n-1} + h_n) \circ \ldots \circ p(h_1,h_2 + h_3 + \ldots + h_n)(f(h_1 + \ldots + h_n)) \\ &= D_{p,f}(h_1,h_2,\ldots,h_n), \end{aligned}$$

by Notation 3.1. This proves (d). Also, by (b), $C_{p,f}$ is collapsable; hence $D_{p,f}(=C_{p,f})$ is collapsable. This proves (c).

(c)⇒(d) : We just proved (b)⇒(d). A similar argument can be used to prove (c)⇒(d).

(d)⇒(b) : By the definition of $C_{p,f}$ and $D_{p,f}$, if $C_{p,f} = D_{p,f}$, then $C_{p,f}$ must be collapsable. Hence (d) implies (b). This completes the proof of the theorem.

**Definition 3.5:** Let $p : H \times H \to P_S$ be a map. We say p is *abelian* if, for all $h_1,h_2,h_3,h_4 \in H$, $p(h_1,h_2) \circ p(h_3,h_4) = p(h_3,h_4) \circ p(h_1,h_2)$.

**Theorem 3.6:** *Let H be an additive group, S be a set, $P_S$ be the set of invertible maps on S, and $p : H \times H \to P_S$ be an abelian map. Then p is an extension function if and only if, for all $h_1,h_2,h_3 \in H$, (3.3) holds; ie,*

$$p(h_1,h_2) \circ p(h_1 + h_2,h_3) = p(h_2,h_3) \circ p(h_1,h_2 + h_3).$$

**Proof** By Lemma 3.3, we need only prove sufficiency, and, by Theorem 3.6, it is sufficient to prove that $C_{p,f} = D_{p,f}$; ie,

$$\prod_{i=2}^{n} p\Big(\sum_{j=1}^{i-1} h_j,h_i\Big) = \prod_{i=1}^{n-1} p\Big(h_i, \sum_{j=i+1}^{n} h_j\Big). \tag{3.5}$$

Now by (3.3),

$$p\Big(\sum_{j=1}^{i} h_j,h_{i+1}\Big) \circ p\Big(\sum_{j=i}^{i+1} h_j, \sum_{j=i+2}^{n} h_j\Big) = p\Big(h_{i+1}, \sum_{j=i+2}^{n} h_j\Big) \circ p\Big(\sum_{j=i}^{i} h_j, \sum_{j=i+1}^{n} h_j\Big).$$

Hence

$$\left[\prod_{i=1}^{n-2} p\Big(\sum_{j=1}^{i} h_j,h_{i+1}\Big)\right] \circ \left[\prod_{i=1}^{n-2} p\Big(\sum_{j=i}^{i+1} h_j, \sum_{j=i+2}^{n} h_j\Big)\right] = \left[\prod_{i=1}^{n-2} p\Big(h_{i+1}, \sum_{j=i+2}^{n} h_j\Big)\right] \circ \left[\prod_{i=1}^{n-2} p\Big(\sum_{j=i}^{i} h_j, \sum_{j=i+1}^{n} h_j\Big)\right].$$

So, because p is an abelian map,

$$\left[\prod_{i=1}^{n-2} p(\sum_{j=1}^{i} h_j, h_{i+1})\right] \circ p(\sum_{j=1}^{n-1} h_j, h_n) = \left[\prod_{i=1}^{n-2} p(h_{i+1}, \sum_{j=i+2}^{n} h_j)\right] \circ p(h_1, \sum_{j=2}^{n} h_j).$$

This simplifies to give (3.5).

**Notes 3.7:** Equation (3.3) has a number of interpretations.

(i) Any bilinear function $p : H \times H \rightarrow P_S$ is an extension function. So (3.3) *may be regarded as a weak bilinearity condition.*

(ii) If $p(H \times H) \subset H \subset P_S$, we may think of p as a multiplication '$\times$' on H which obeys, with respect to group addition, a weak left and right distributivity; viz.,

$$h_1 \times h_2 + (h_1 + h_2) \times h_3 = h_2 \times h_3 + h_1 \times (h_2 + h_3). \tag{3.6}$$

Indeed, *any ring muliplication on* $H \subset P_S$ *defines an extension function, and any extension function* p *where* $p(H \times H) \subset H \subset P_S$ *defines a weak (possibly non-associative) ring multiplication on* H.

(iii) *The relation* (3.3) *is equivalent to* $|H|^3$ *linear equations involving the* $|H|^2$ *quantities* $p(h_1,h_2)$ $(h_1,h_2 \in H)$ *over the integers*. So, in principle, we can use linear algebra to find all abelian extension functions.

Whenever H is cyclic, we are able to determine all of the abelian extension functions $p : H \times H \rightarrow P_S$.

**Theorem 3.8:** *Let* S *be any set, and let* H *be the additive cyclic group of order* v. *If* $p : H \times H \rightarrow P_S$ *is an abelian extension map, then, for all* $s,t \in H$,

$$p(s,t) = p(0,0) \circ \left[\prod_{i=0}^{t-1} (p(i,1)^{-1} \circ p(s+i,1))\right]. \tag{3.7}$$

*Conversely, if* $A_0, A_1, \ldots, A_{v-1}$ *are commuting invertible maps on* S, *which are indexed by the elements of* H, *then* $q : H \times H \rightarrow P_S$ *where*

$$q(s,t) = A_0 \circ \left[\prod_{i=0}^{t-1} (A_i^{-1} \circ A_{s+i})\right],$$

*is an extension of a function.*

**Proof** By (3,1), equation (3.7) is true for t=0. Now suppose it is true for $0 \le t \le n$; we show it applies when t = n+1. By (3.3) (with $h_1 = s$, $h_2 = n$ and $h_3 = 1$), we have

$$p(s,n+1) = p(s,n) \circ p(n,1)^{-1} \circ p(s+n,1).$$

Now use the formula to expand p(s,n) to obtain the result for p(s,n+1). This proves (3.7).

To prove the second part of the theorem, it is sufficient to verify that q satisfies (3.3). First we note that, if $t' \equiv t \pmod v$, then

$$\prod_{i=0}^{t'-1} A_i^{-1} \circ A_{s+i} = \prod_{i=0}^{t-1} A_i^{-1} \circ A_{s+i}. \tag{3.8}$$

This observation allows us to do the following calculation.

$$\begin{aligned} q(s,t) \circ q(s+t,u) &= A_0 \circ \left[\prod_{i=0}^{t-1} (A_i^{-1} \circ A_{s+i})\right] \circ A_0 \circ \left[\prod_{i=0}^{u-1} (A_i^{-1} \circ A_{s+t+i})\right] \\ &= A_0 \circ A_0 \circ \left[\prod_{i=0}^{t-1} (A_i^{-1} \circ A_{s+i})\right] \circ \left[\prod_{i=0}^{u-1} (A_i^{-1} \circ A_{t+i})\right] \circ \left[\prod_{i=0}^{u-1} (A_{t+i}^{-1} \circ A_{s+t+i})\right] \\ &= (A_0 \circ \left[\prod_{i=0}^{u-1} (A_i^{-1} \circ A_{t+i})\right]) \circ (A_0 \circ \left[\prod_{i=0}^{u+t-1} (A_i^{-1} \circ A_{s+i})\right]) \\ &= q(t,u) \circ q(s,t+u). \end{aligned}$$

This verifies (3.3), and completes the proof of the theorem.

We note that (3.8) allows us to define extension functions over the additive group of integers via (3.7).

**Theorem 3.9:** *Suppose* $p : H \times H \to P_S$ *and* $q : K \times K \to P_S$ *are abelian extension functions. Define* $p \oplus q : (H \oplus K) \times (H \oplus K) \to P_S$ *by*

$$p \oplus q((h_1,k_1), (h_2,k_2)) = p(h_1,h_2) \circ q(k_1,k_2),$$

*for all* $h_1,h_2 \in H$ *and* $k_1,k_2 \in K$. *If the maps* $p(h_1,h_2)$ *and* $q(k_1,k_2)$ *commute for all* $h_1,h_2 \in H$ *and* $k_1,k_2 \in K$, *then* $p \oplus q$ *is an abelian extension function over* $H \oplus K$.

**Proof** Immediate from (3.3) and Theorem 3.6.

Theorems 3.8 and 3.9 ensure that for any abelian group, there are many extension functions: any abelian group H may be written as a direct sum of cyclic groups, and (3.7) can be applied independently to each cyclic component as in Theorem 3.9. Example 3.10, below, indicates that, if H is not cyclic, not all extension functions can be obtained in this way.

**Example 3.10:** Let H = {e,a,b,ab} be the elementary abelian group of order 4 ($\mathbf{Z}_2 \times \mathbf{Z}_2$). By solving the 64 equations referred to in Notes 3.7(iii), we obtained the following "muliplication" table for the general extension function over H.

p(s,t) :

| s \ t | e | a | b | ab |
|---|---|---|---|---|
| e | Z | Z | Z | Z |
| a | Z | A | $E^{-1}$ | AZE |
| b | Z | $B^{-1}$ | F | BZF |
| ab | Z | ABZ | EFZ | AEFZB |

where A,B,E,F and Z are many commuting invertible maps such that $E \circ E = B \circ B$.

## 4. DESIGNS AND EXTENSION FUNCTIONS

In this section, we use extension functions to construct proper n-dimensional designs. We will restrict our attention to designs with $\Pi_R = \Pi_C = \Pi$.

**Definition 4.1:** Let X be a $(v,\Pi,\beta,S)$-design. We say X *has an extension function* $p : H \times H \to P_S$ if p is an extension function, and there exists a map $f : H \to S$ such that $X = (p(h_1,h_2)(f(h_1 + h_2)))$. If $p(H \times H) \in \Pi$, we say p is *suitable.*

**Theorem 4.2:** *Let* X *be a* $(v,\Pi,\beta,S)$*-design with a suitable extension function* $p : H \times H \to S$ *such that* $X = (p(h_1,h_2)(f(h_1,h_2)))$. *Then* $(C_{p,f}(h_1,h_2,...h_n))$ *is a proper n-dimensional* $(v,\Pi,\beta,S)^n$*-design.*

**Proof** By Definition 3.2, $C_{p,f}$ is collapsable. So by Theorem 2.9, $(C_{p,f}(h_1,h_2,...h_n))$ is a proper $(v,\Pi,\beta,S)^n$-design.

**Example 4.3:** In this example, we index certain 4 x 4 designs with the elements, e,a,b,ab of $\mathbf{Z}_2 \times \mathbf{Z}_2$, and we show how these designs all have suitable extension functions. Now, Example 3.10 gives the multiplication table for the general extension function over $\mathbf{Z}_2 \times \mathbf{Z}_2$. To specify an extension function over $\mathbb{Z}_2 \times \mathbb{Z}_2$, it is therefore sufficient to specify A,B,E,F, and Z. We do this for each of the designs below.

(i) A Hadamard matrix of order 4.
$f(e) = f(a) = f(b) = f(ab) = 1$;
B,E and F are multiplication by $-1$,
and A and Z are the identity.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & - & 1 \\ 1 & - & 1 & - \end{pmatrix}$$

(ii) A generalised Hadamard matrix of order 4.
$f(e) = f(a) = f(b) = f(ab) = e$,
A,B,E,F and Z are multiplication
by a,ab,b,a, and e respectively.

$$\begin{pmatrix} e & e & e & e \\ e & a & b & ab \\ e & ab & a & b \\ e & b & ab & a \end{pmatrix}$$

(iii) Both orthogonal designs of type (1,1,1,1) and order 4. Here $S = \{\pm a, \pm b, \pm c, \pm d\}$.

(a) $f(e) = a$, $f(a) = b$, $f(b) = c$, $f(ab) = d$;
A,B,E,F and Z are multiplication
by $-1,1,-1,-1$, and 1 respectively.

$$\begin{pmatrix} a & b & c & d \\ b & -a & -d & c \\ c & d & -a & -b \\ d & -c & b & -a \end{pmatrix}$$

(b) f as above; A,B,E,F and Z are
multiplication by $-1,1,-1,-1$,
and 1 respectively.

$$\begin{pmatrix} a & b & c & d \\ b & -a & d & c \\ c & -d & -a & b \\ d & c & -b & -a \end{pmatrix}$$

All of these extension maps are suitable. Hence, Theorem 4.2 may be used to construct proper n-dimensional designs of the various sorts.

**Example 4.4:** If X is based on a difference set, then, for some group H and some map $f : H \to S$, $X = (f(h_1 + h_2))$ $(h_1,h_2 \in H)$. So the trivial extension function $p : H \times H \to P_S$ (where $p(h_1,h_2)$ is the identity map for all $h_1,h_2 \in H$) is a suitable extension function for X. Thus our construction generalises Hammer and Seberry's construction.

Let q be a prime power, and let EA(q) denote the elementary abelian group of order q. Let X be a muliplication table of GF(q) (ie, $X = (h_1h_2)$ $(h_1,h_2 \in EA(q))$). Drake (1979) showed that X is a generalised Hadamard matrix (written GH(q;EA(q)) of order q over the additive group, EA(q), of GF(q).

**Theorem 4.5:** *The* GH(q;EA(q)), $X = (h_1h_2)$ $(h_1,h_2 \in EA(q))$ *has a suitable extension function over* EA(q).

**Proof** Let $\pi_h$:EA(q) $\rightarrow$EA(q) be multiplication by h. X is a (q,$\Pi$,$\beta$,EA(q))-design with $\Pi = \{\pi_h \mid h \in EA(q)\}$. Define p:EA(q) $\times$ EA(q) $\rightarrow P_{EA(q)}$ so that, for all $h_1,h_2 \in H$, $p(h_1,h_2) = \pi_{h_1h_2}$. So $p(h_1,h_2) \in \Pi$ for all $h_1,h_2 \in EA(q)$. Also p is a ring multiplication over an isomorphic image of EA(q) contained in $P_{EA(q)}$; so, by Notes 3.7 (ii), p is an extension function. But $X = (p(h_1,h_2)(f(h_1 + h_2)))$, where f(h) = e for all $h \in H$; so p is a suitable extension function for GH(q;EA(q)).

We show that negacyclic weighing matrices and $\omega$-cyclic generalised weighing matrices may be used in Theorem 4.2 to give proper n-dimensional designs of a particularly simple form.

**Thorem 4.6:** *Let* $X_2$ *be a back negacyclic weighing matrix or a back* $\omega$*-circulant generalised weighing matrix; then* $X_2$ *has a suitable extension function. Hence, if there exists a negacyclic* W(v,k) *matrix or an* $\omega$*-circulant* GW(v,k;G), *then there exists a proper* n*-dimensional design of the same sort. Indeed, there exists a periodic function* $F:Z \rightarrow \{-1,0,1\}$ *(respectively* $F:Z \rightarrow \{0\} \cup G$) *such that* $X_n = (F(s_1 + s_2 + s_3 + .... + s_n))$ $(0 \le s_i \le v-1)$.

Proof: A $v \times v$ matrix is back $\omega$-circulant if the (i+1)th row of A is

$$a_i,\ a_{i+1},\dots,a_{v-1},\omega a_0,\omega a_1,\dots,\omega a_{i-1},$$

where $a_1, a_2,\dots,a_v$ is the first row of A. Now, in (3.7) put p(1,1) = p(2,1) =.... p(v-2,1) = $\pi_e$, and put p(v-1,1) = $\pi_\omega$. Also, put $f(s+t) = a_{s+t}$. Then $A = p(s,t) \circ f(s+t)$. Also, $\pi_\omega \in \Pi$; so p is suitable. Hence $X_n = (C_{p,f}(h_1,h_2,\dots,h_n))$is a proper n-dimensional design. This proves the first part of the theorem for $\omega$-circulant matrices. To prove this part for negacyclic matrices put $\omega = -1$.

Now we prove the second part of the theorem. Let v be a positive integer, and let a be any integer. Let $\{a\}_v$ be the integer $x \equiv a \pmod v$ such that $0 \le x \le v-1$, and let $[a]_v$ be the greatest integer not greater than a/v. Then

$$h_1 + h_2 + h_3 = v[h_1 + h_2]_v + \{h_1 + h_2\}_v + h_3;$$

indeed,

$$\sum_{i=1}^{n} h_i = v[h_1 + h_2]_v + v[\{h_1 + h_2\}_v + h_3]_v + v[\{h_1 + h_2 + h_3\}_v + h_4]_v$$

$$+\dots+ [\{\sum_{i=1}^{n-2} h_i\}_v + h_{n-1}]_v + \{h_1 + \dots + h_{n-1}\}_v + h_n.$$

So

$$[\sum_{i=1}^{n} h_i]_v = [h_1 + h_2]_v + [\{h_1 + h_2\}_v + h_3]_v + [\{h_1 + h_2 + h_3\}_v + h_4]_v + \ldots$$

$$+ [\{h_1 + h_2 + \ldots + h_{n-1}\}_v + h_n]_v.$$

Hence,

$$p(1,v-1)^{[\sum_{i=1}^{n} h_i]_v} = p(h_1,h_2) \circ p(h_1 + h_2,h_3) \circ \ldots \circ p(h_1 + h_2 + \ldots + h_{n-1},h_n).$$

So

$$C_{p,f}(h_1,h_2 ,\ldots, h_n) = p(1,v-1)^{[\sum_{i=1}^{n} h_i]_v} (f(h_1 + h_2 + \ldots + h_n)).$$

Now define $F:\mathbf{Z}\rightarrow\{0\}\cup G$ so that, for all $s\in\mathbf{Z}$,

$$F(s) = \omega^{[s]_v} f(\{s\}_v);$$

then $X_n = (F(s_1 + s_2 + \ldots + s_n))$ $(0 \le s_i \le v-1, s_i\in \mathbf{Z})$. We note that, if $\omega$ has order m, then F has period mv. For negacyclic matrices put $\omega = -1$ and $G = \{-1,1\}$. In this case $m = 2$. This completes the proof.

Now suppose $X_2$ is negacyclic, and consider the map $F_2 : \mathbf{Z}\times\mathbf{Z}\rightarrow\{-1,0,1\}$ where $F_2(h_1,h_2) = F(h_1 + h_2)$. We can obtain an infinite 2-dimensional "checkerboard" pattern of coloured cells if we, say, colour the (i,j)th cell white, red or black according to whether $F_2(i,j) = -1, 0$ or $1$. The faces of $X_n$ can then be obtained by sliding a $v \times v$ window around the pattern. Shlicta (1979) noted that some of his higher dimensional Hadamard matrices had this "periodic" property. Theorem 4.6 provides examples for various types of designs including conference matrices.

Finally, we remark that Theorem 3.9 may be used to generate suitable extension functions for the Kronecker product of (generalised) weighing matrices which have suitable extension functions. We demonstrate this with a theorem and an example.

**Theorem 4.7:** *Let* $F : \mathbf{Z}^t\rightarrow\{-1,1\}$ *be the map such that, for all* $i_1,i_2,\ldots,i_t\in \mathbf{Z}$,

$$F(i_1,i_2,\ldots,i_t) = \prod_{r=1}^{t} (-1)^{[i_r]_2}$$

*Then* $(F((i_{11},i_{12},\ldots,i_{1t}) + (i_{21},i_{22},\ldots,i_{2t}) + \ldots + (i_{n1},i_{n2},\ldots,i_{nt})))$ $(0 \le i_{jk} \le 1)$ *is a proper n-dimensional Hadamard matrix of order* $2^t$.

**Proof** The initial row 1, 1 generates a negacyclic Hadamard matrix $H_2$ of order 2. The negacyclic extension function with v = 2 can be used to extend this design to n-dimensions. Now apply Theorem 3.9 t–1 times to obtain the direct product extension function for

$H_{2^t} = H_2 \times H_2 \times \ldots \times H_2$ (t times). The argument in the proof of Theorem 4.6 applies to each component of $(i_1, i_2, \ldots, i_t)$.

**Example 4.8:** Let J be the $3 \times 3$ matrix whose entries are all 1. The following Hadamard matrix of order 12 was obtained by substituting a set of Williamson matrices of the order 3 (namely: J, 2I–J, 2I–J, 2I–J) for the indeterminants a, b, c and d in the orthogonal design in Example 4.3(iii)(a).

$$H = \begin{pmatrix} J & 2I-J & 2I-J & 2I-J \\ 2I-J & -J & J-2I & 2I-J \\ 2I-J & 2I-J & -J & J-2I \\ 2I-J & J-2I & 2I-J & -J \end{pmatrix}$$

Let $EA(4) = \mathbf{Z}_2 \times \mathbf{Z}_2 = \langle a,b \mid ab = ba,\ a^2 = b^2 = e\rangle$, and let $\mathbf{Z}_3 = \langle \omega \mid \omega^3 = e\rangle$.
Define $f : EA(4) \times \mathbf{Z}_3 \rightarrow \{-1,1\}$ so that

$$f(x) = \begin{cases} -1 & \text{for } x = a\omega, a\omega^2, b\omega, b\omega^2, ab\omega, ab\omega^2 \\ 1 & \text{otherwise} \end{cases}$$

Finally, let p(s,t) be the extension function defined in Example 4.3(iii)(a), and, let q be the trivial extension function over $\mathbf{Z}_3$ (For all $s,t \in \mathbf{Z}_3$, q(s,t) is the identity map on {1,-1}). Now, index H with the elements of the group $EA(4) \times \mathbf{Z}_3$ in the following order: $e, \omega, \omega^2, a, a\omega, a\omega^2, b, b\omega, b\omega^2, ab, ab\omega, ab\omega^2$. Then

$$H = (p \oplus q(s,t)(f(s+t)))\ (s,t \in EA(4) \times \mathbf{Z}_3),$$

and $p \oplus q$ is a suitable function of H.

Acknowledgement The author would like to thank J. Hammer for a number of enjoyable and stimulating discussions about higher dimensional designs during the early development of the ideas of this paper.

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