GENERALIZED BHASKAR RAO DESIGNS WITH TWO ASSOCIATION CLASSES

William D. PALMER
Department of Pure Mathematics
The University of Sydney
and Department of Mathematics
Sydney Institute of Education

Abstract. In previous work generalized Bhaskar Rao designs whose underlying design is a balanced incomplete block design have been considered. In the first section of this paper generalized Bhaskar Rao designs (with 2 association classes) whose underlying design is a group divisible design are defined. Some methods for the construction of these designs are developed in the second section. It is shown that the necessary conditions:

$$\begin{aligned} \lambda &\equiv 0 \pmod{g} \\ v &\equiv 0 \pmod{2} \\ v &\geq 6 \\ \lambda v(v-2) &\equiv 0 \pmod{3} \end{aligned}$$

are sufficient for the existence of a GBRD($v,3,\lambda,2$;EA(g)) where EA(g) is an elementary abelian group of order g. Finally, the design GBRD($v,b,r,3,\lambda_1 = 0,\lambda_2 = \lambda,2$;EA($g$)) is used to construct a group divisible design with $v/2$ groups each of size $2g$ and with the parameters

$v^* = vg,\ b^* = bg,\ r^* = r,\ k^* = 3,\ \lambda_1{}^* = 0,\ \lambda_2{}^* = \lambda/g.$

1. Introduction

A *design* is a pair (X,R) where X is a finite set of elements and R is a collection of (not necessarily distinct) subsets R_i (called *blocks*) of X.

A *balanced incomplete block design,* BIBD(v,b,r,k,λ), is an arrangement of v elements into b blocks such that:

(i) each element appears in exactly r blocks;

(ii) each block contains exactly $k(<v)$ elements; and

(iii) each pair of distinct elements appear together in exactly λ blocks.

As $r(k-1) = \lambda(v-1)$ and $vr = bk$ are well-known necessary conditions for the existence of a BIBD(v,b,r,k,λ) we denote this design by BIBD(v,k,λ).

Let v and λ be positive integers and K a set of positive integers. An arrangement of the element of a set X into blocks is a *pairwise balanced design*, PBD$(v;K;\lambda)$, if:

(i) X contains exactly v elements;

(ii) if a block contains k elements then k belongs to K;

(iii) each pair of distinct elements appear together in exactly λ blocks.

A pairwise balanced design PBD$(v;\{k\};\lambda)$, where $K = \{k\}$ consists of exactly one integer, is a BIBD(v,k,λ).

A *group divisible design*, GDD, on v points is a triple (X,S,A) where:

(i) X is a set (of *points*);

(ii) S is a class of non-empty subsets of X (called *groups*) which partition X;

(iii) A is a class of subsets of X (called *blocks*), each containing at least two points;

(iv) each pair $\{x,y\}$ of points contained in a group is contained in exactly λ_1 blocks;

(v) each pair $\{x,y\}$ of points not contained in a group is contained in exactly λ_2 blocks.

We say that GDD($v,b,r,\lambda_1,\lambda_2,m,n$) is a GDD with m groups each of size n, where all b blocks have size k and each point lies in r blocks. In this paper we are concerned with the class of GDD's with $\lambda_1 = 0$ and $\lambda_2 = \lambda$; and a GDD in this class will be denoted in terms of the independent parameters v,k,λ,n by GDD(v,k,λ,n).

Suppose that x and y are distinct points in the GDD(v,k,λ,n). We say that x and y are first associates if $\{x,y\}$ is contained in a group. If $\{x,y\}$ is not contained in a group then x and y are said to be second associates. We define the *association matrices*

$$R_i = \left(b^i_{st}\right), \ 1 \le i \le 2, \text{ and } 1 \le s, \ t \le v$$

of a GDD(v,k,λ,n) as $v x v$ (0,1) matrices given by

$$\begin{aligned} b^i_{st} &= 1, \quad \text{if } s \text{ and } t \text{ are ith associates,} \\ &= 0, \text{ otherwise.} \end{aligned}$$

Let $G = \{h_1 = e, h_2,\ldots,h_g\}$ be a finite group (with identity e) of order g.
Form the matrix W,

$$W = h_1A_1+\ldots + h_gA_g$$

where $A_1,\ldots,A_g$ are vxb (0,1) - matrices such that the Hadamard product $A_k*A_j = 0$ for any $k\neq j$. Let

$$W^+ = (h_1{}^{-1}A_1+\ldots+h_g{}^{-1}A_g)^T,$$

and $$N = A_1 + A_2 + A_3+\ldots+ A_g$$

Then we say that W is a *partial generalized Bhaskar Rao design with two association classes* over G denoted by PGBRD($v,b,r,k,\lambda,n;G$), or in abbreviated form PGBRD($v,k,\lambda,n;G$), if N satisfies

$$NN^T = rI + \lambda R_2; \quad (1.1)$$

that is, N is the incidence matrix of the GDD(v,k,λ,n), and

$$WW^+ = reI + (\lambda/g)(h_1+\ldots+h_g)R_2 \quad (1.2)$$

A generalized Bhaskar Rao design with one association class, denoted by GBRD($v,k,\lambda;G$), satisfies

$$NN^T = (r-\lambda)I + \lambda J; \quad (1.3)$$

that is, if $k < v$, N is the incidence matrix of the BIBD(v,b,r,k,λ), and

$$WW^+ = reI + (\lambda/g)(h_1+\ldots+h_g)(J-I). \quad (1.4)$$

In both cases we say that the design W is based on the matrix N.

We shall reserve the name generalized Bhaskar Rao design for a generalized Bhaskar Rao design with one association class.

A generalized Bhaskar Rao design with one association class and $v = b$ is a symmetric GBRD or a *generalized weighing matrix*. A generalized weighing matrix which contains no zero entries is also known as a *generalized Hadamard matrix*. Generalized Hadamard matrices have been studied by Brock (1988), Dawson (1985), and de Launey (1984, 1986, 1987, 1989A, 1989B), Jungnickel (1979), Seberry (1979), Street (1979).

Generalized Bhaskar Rao designs with one association class over elementary abelian groups other than Z_2 have been studied recently by Lam and Seberry (1984) and Seberry (1985). de Launey, Sarvate and Seberry (1985) considered generalized Bhaskar Rao designs over Z_4 which is an abelian (but not elementary) group. Some generalized Bhaskar Rao designs over various groups (abelian and non-abelian) have been studied by Gibbons and Mathon (1987A, 1987B). Palmer and Seberry (1988) have shown that the necessary conditions are sufficient for the existence of generalized Bhaskar Rao designs over the non-abelian groups S_3, D_4, Q_4, D_6 and over the abelian group $Z_2 \times Z_4$. Seberry (1988) has considered generalized Bhaskar Rao designs over the cyclic group Z_8. Recently, Curran and Vanstone (1989) have used generalized Bhaskar Rao designs to construct doubly resolvable BIBDs. Generalized Bhaskar Rao designs and generalized Hadamard matrices have been used by Mackenzie and Seberry (1988) to obtain q-ary codes.

In this paper we are concerned with the existence of the designs PGBRD(v,3,λ,2;EA(g)). The *elementary abelian group* EA(g) is defined to be the abelian group of order g so that every element of EA(g) has prime order. EA(g) is the direct product of the groups $(Z_p)^i$ where p is the prime of multiplicity i in the primary decomposition of g.

Example 1.1. The matrix W given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & 0 & 0 & 0 & 1 & w & w^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & w & w^2 & 0 & 0 & 0 & 1 & w & w^2 \\ 1 & w^2 & w & 0 & 0 & 0 & 0 & 0 & 0 & 1 & w^2 & w \\ 0 & 0 & 0 & 1 & w^2 & w & 1 & w^2 & w & 0 & 0 & 0 \end{bmatrix}$$

is a PGBRD(6,3,3,2,Z_3) based on the 6x12 incidence matrix of a GDD(6,3,3,2) which is a 3-multiple of design SR18 recorded by Clatworthy (1973, p.144).

2. Constructions

In this section we establish some general constructions which will be used extensively in the remaining sections of the paper.

Theorem 2.1 *Suppose that there exists a* GDD(v,k,λ,n) *and that there exists a* GBRD$(k,k,\mu;G)$. *Then a* PGBRD$(v,k,\lambda\mu,n,G)$ *exists.*

Proof: Let N be the incidence matrix of a GDD(v,k,λ,n) and suppose that e_i, $i = 1,\ldots,k$ are distinct rows of a GBRD$(k,k,\mu;G)$. We form a matrix W by replacing the 1's of each column of N by the vectors $\mathbf{e}_i$. The zeros of N are replaced by $(0,0,\ldots,0)$. The matrix W is a PGBRD$(v,k,\lambda\mu,n;G)$.

Example 2.2.

$$\begin{bmatrix} e & e & 0 & 0 \\ 0 & 0 & e & e \\ f & 0 & f & 0 \\ 0 & f & 0 & f \\ h & 0 & 0 & h \\ 0 & h & h & 0 \end{bmatrix}$$

is a PGBRD$(6,3,g,2;G)$ where **e**, **f**, and **h** are three distinct rows of a GH(g,G). This PGBRD$(6,3,g,2;G)$ is obtained by replacing the 1's of the incidence matrix of a GDD(6,3,1,2) by **e**, **f**, and **h** in the manner described in the proof of Theorem 2.1.

Example 1.1 was constructed by application of Theorem 2.1.

Corollary 2.3. *Suppose that $v \geq 6$ and $v \equiv 0$ or $2 \pmod 6$. Then:*

(i) a PGBRD($v,3,g,2$;EA(g)) *exists if $g \equiv 0$ or 1 or $3 \pmod 4$*

(ii) a PGBRD($v,3,g,2;G$) *exists if G is an abelian group of odd order, g.*

Proof: Hanani (1975, p. 355) has shown that a GDD(v,3,1,2) exists if and only if $v \geq 6$ and $v \equiv 0$ or 2 (mod 6). A GBRD(3,3,g;EA(g)) exists if and only if $g \equiv 0$, 1 or 3 (mod 4) (Seberry (1985)). Likewise, if G is an abelian group of odd order, g then a GBRD(3,3,$g;G$) exists (Lam and Seberry (1984)). The results follow on application of Theorem 2.1.

Theorem 2.4. *Suppose that G is a group of order g such that a* GBRD$(v,k,2g;G\times Z_2)$ *exists. Then a* PGBRD$(2v,k,g,2;G)$ *exists.*

Proof: Let $G = \{e=h_1,h_2,\ldots,h_g\}$, I_2 be the identity matrix of size 2, and J_2 be the square matrix of size 2 whose entries are all 1's. We replace each zero entry in the GBRD($v,k,2g;G\times Z_2$) by a square zero matrix of size 2 and the non-zero entries are replaced in the following manner:

the entry $+h_i$ is replaced by h_iI_2 and

the entry $-h_i$ is replaced by $h_i(J_2-I_2)$. The matrix thus formed is a PGBRD($2v,k,g,2;G$).

Seberry (1985) has shown that the necessary conditions are sufficient for the existence of a GBRD($v,3,tg$;EA(g)) over the elementary abelian group EA(g). Hence, a GBRD($V,3,2tg;Z_2\times$EA(g)) exists if and only if $V \geqslant 3$ and

$$tg\,V(V-1) \equiv 0 \pmod{12}.$$

The following table summarises the constraints on the triples t, g and v when tg is even.

tg (mod 6)	Constraint on $v \geqslant 3$
$\equiv 0$	none
$\equiv 2$ or 4	$v \equiv 0$ or 1 (mod 3)

Hence, by Theorem 2.4, we have

Corollary 2.5. *There exists a* PGBRD($v,3,tg,2$;EA(g)) *whenever*

$tg \equiv 0 \pmod 6$, $\quad v = 6,8,10,\ldots$

$tg \equiv 2$ *or* $4 \pmod 6$, $\quad v \geq 6$ *and* $v \equiv 0$ *or* $2 \pmod 6$.

Lemma 2.6. (Hanani (1975, Lemma 2.16)). *Suppose that a* PBD($v;H;\lambda$) *exists and that there exist* GDD(mh,k,μ,m) *for each h belonging to H, then a* GDD($mv,k,\lambda\mu,m$) *exists.*

Lemma 2.7. (Hanani (1975, Lemma 5.3)). *For every $v \geq 3$, the* PBD($v;K_3;1$) *exists where* $K_3 = \{3,4,5,6,8\}$.

Theorem 2.8. *Let G be a finite group. Suppose that a* PBD($v;H;\lambda$) *exists and that for each h belonging to H a* PGBRD($mh,k,\mu,m;G$) *exists. Then a* PGBRD($mv,k,\lambda\mu,m;G$) *exists.*

Proof: Let N be an incidence matrix for a PBD($v;H;1$). We form a matrix W from N in the following manner. Suppose the first column of N contains h 1's. We then partition the matrix A^h =PGBRD($mh,k,\mu,m;G$) as follows:

$$\begin{bmatrix} A_1{}^h \\ A_2{}^h \\ \cdots \\ A_h{}^h \end{bmatrix}$$

where A_i^h, $1 \le i \le h$ are matrices each consisting of m distinct rows of A^h. We then replace the first 1 in the first column of N by A_1^h, the second 1 in the first column of N by A_2^h, and so on. The zeros in the first column of N are replaced by a zero matrix of the size as each A_i^h.

The process is repeated for the remaining columns of N to construct the matrix W. We claim that W is a PGBRD($mv,k,\lambda\mu,m;G$).

Example 2.9. The incidence matrix of a PBD(11;{3,5};1) (Hanani (1975, p. 289)) is exhibited below:

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}$$

We construct a PGBRD(22,3,3,2;Z_3) by replacing the 1's in all columns except the last by 2x12 matrices coming from the partitioning of the PGBRD(6,3,3,2;Z_3) which was constructed in Example 2.2. The 1's in the last column are replaced by the 2x40 matrices arising from the partitioning of a design PGBRD(10,3,3,2;Z_3) which exists by application of Theorem 2.4 to a GBRD(5,3,6;$Z_3 \times Z_2$). The 0's in the first 15 columns are replaced by a 2x12 zero matrix and the remaining 0's are replaced by a 2x40 zero matrix.

Theorem 2.10. *Suppose that there exists a* BIBD(v,k,λ) *and a* PGBRD($mk,j,\mu,m;G$). *Then there exists a* PGBRD($mv,j,\lambda\mu,m;G$).

Proof: We observe that a BIBD(v,k,λ) is a PBD($v,\{k\},\lambda$) and then apply Theorem 2.8.

Corollary 2.11. *Let G be a group of order g. There exists a* PGBRD($10,3,3g,2;G$) *whenever there exists a* PGBRD($6,3,g,2;G$).

Proof: A BIBD(5,3,3) exists by Hanani (1975, Theorem 5.1). The result follows from Theorem 2.10 with the BIBD(5,3,3) and the PGBRD($6,3,g,2;G$).

Corollary 2.12. *Suppose that G is a group of order g. There exists a* PGBRD($16,3,3g,2;G$) *whenever there exists a* PGBRD($8,3,g,2;G$).

Proof: A BIBD(8,4,3) exists by Hanani (1975, Theorem 5.2). This design used with PGBRD($8,3,g,2;G$) produces a PGBRD($16,3,g,2;G$).

Theorem 2.13. *Suppose that there exists a* GBRD($v,k,\lambda;G$), *A, and a* PGBRD($mk,j,\mu,m;H$), *R. Then there exists a* PGBRD($mv,j,\lambda\mu,m;GxH$).

Proof: The new PGBRD is obtained by replacing the jth non-zero entry, say x, of each column of A by x times R_i, where R_i is a matrix with m rows in the partitioning of R given by

$$\begin{bmatrix} R_1 \\ R_2 \\ \cdots \\ R_i \\ \cdots \\ R_k \end{bmatrix}$$

The next theorem is an adaptation of a construction of GBRD/1s given in Gibbons and Mathon (1987, p.12).

Theorem 2.14. *Let W be a PGBRD$(v,k,\lambda,n;G)$ and suppose that G contains a normal subgroup T. Then there exists a PGBRD$(v,k,\lambda,n;H)$, where $H = G/T$ is the factor group of G with respect to T.*

Proof: In W, replace each group element x (say) by its coset Tx. The new matrix thus formed is a GBRD$(v,k,\lambda,n;H)$ where $H = G/T$.

3. Necessary Conditions

Hanani (1975, p.355) has shown that the necessary conditions are sufficient for the existence of a GDD(v,k,λ,n) where $k = 3$. These conditions are

$$v \equiv 0 \pmod{n} \tag{3.1}$$
$$v \geq 3n \tag{3.2}$$
$$\lambda(v-n) \equiv 0 \pmod{2} \tag{3.3}$$
$$\lambda v(v-n) \equiv 0 \pmod{6} \tag{3.4}$$

For the existence of a PGBRD$(v,k,\lambda,n;G)$ we also require

$$\lambda \equiv 0 \pmod{g} \tag{3.5}$$

where g is the order of the group G.

We now give an extra necessary condition for the existence of a PGBRD/2 over Z_2. This condition is an adaptation of Theorem 1 of Seberry (1984) which is concerned with GBRDs over Z_2.

Theorem 3.1. *A* PGBRD$(v,k,\lambda,n;Z_2)$, *W, can only exist if the equations:*

(i) $x_3 + 3x_5 + 6x_7 + \ldots + ((k^2-1)/8)x_k = b(k-1)/8$
for k *odd,*

(ii) $-x_0 + 3x_4 + 8x_6 + \ldots + ((k^2-4)/4)x_k = b(k-4)/4$
for k *even,*

have integral solutions. In particular, a PGBRD$(v,3,\lambda,n;Z_2)$ *can only exist if* $b \equiv 0 \pmod 4$.

In view of Theorem 2.14 we see that if g is even the existence of a PGBRD$(v,k,\lambda,n;\mathrm{EA}(g))$ would imply the existence of a PGBRD$(v,k,\lambda,n;Z_2)$. Thus, by Theorem 3.1,

$$b = \frac{\lambda v(v-n)}{6} \equiv 0 \pmod 4$$

or

$$\lambda v(v-n) \equiv 0 \pmod{24} \qquad (3.6)$$

is a necessary condition for the existence of a PGBRD$(v,3,\lambda,n;\mathrm{EA}(g))$ when g is even. Hence, we have

Theorem 3.2. *The necessary conditions for the existence of a* PGBRD$(v,3,\lambda = tg,2;\ \mathrm{EA}(g))$ *are:*

$$\lambda \equiv 0 \pmod g \qquad (3.7)$$

$$v \geq 6 \qquad (3.8)$$

$$v \equiv 0 \pmod 2 \qquad (3.9)$$

$$\lambda v(v-2) \equiv 0 \pmod 3 \qquad (3.10)$$

In the remaining parts of the paper we will show that the necessary conditions are sufficient for the existence of a PGBRD$(v,3,\lambda=tg,2;\mathrm{EA}(g))$.

4. Existence of PGBRD over EA(g), g odd

Theorem 4.1. *If $g \equiv 1$ or $5 \pmod 6$, the necessary conditions (3.7), (3.8), (3.9) and (3.10) are sufficient for the existence of a* PGBRD($v,3,\lambda=tg,2;EA(g)$).

Proof: The necessary conditions for the existence of a PGBRD($v,3,\lambda=tg,2;$EA(g)), $g \equiv 1$ or $5 \pmod 6$ are:

$t \equiv 0 \pmod 3$, $v = 6, 8, 10 \ldots$

$t \equiv 1$ or $2 \pmod 3$, $v \geq 6$ and $v \equiv 0$ or $2 \pmod 6$.

We consider two cases.

Case 1: $\lambda = g$. Corollary 2.3 tells us a PGBRD($v,3,g,2;$EA(g)) exists when $v \geq 6$ and $v \equiv 0$ or 2 (mod 6). By taking t copies of this design we can construct a PGBRD($v,3,\lambda=tg,2;$EA(g)).

Case 2: $\lambda = 3g$. By Lemma 2.7 and Theorem 2.8 we need only establish the existence of a PGBRD($v,3,3g,2;$EA(g)) for v belonging to $\{6,8,10,12,16\}$ to show there exists a PGBRD($u,3,3g,2;$EA(g)) for all u belonging to $\{6,8, \ldots\}$.

The designs PGBRD($v,3,3g,2;$EA(g)) exist for

v	Reason
6	PGBRD($6,3,3g,2;$EA(g)) is 3 copies of PGBRD($6,3,g,2;$EA(g)), given in Case 1.
8	PGBRD($8,3,3g,2;$EA(g)) is 3 copies of PGBRD($8,3,g,2;$EA(g)), given in Case 1.
10	Apply Corollary 2.11 to the PGBRD($6,3,g,2;$EA(g)).
12	PGBRD($12,3,3g,2;$EA(g)) is 3 copies of PGBRD($12,3,g,2;$EA(g)), given in Case 1.
16	Apply Corollary 2.12 to the PGBRD($8,3,g,2;$EA(g)).

By taking t copies of the newly constructed designs PGBRD(u,3,3g,2;EA(g)), u belonging to {6,8,10,...}, we can produce a PGBRD(u,3,3tg,2;EA(g)).

Theorem 4.2. *If $g \equiv 3 \pmod 6$ the necessary conditions (3.7), (3.8), (3.9) and (3.10) are sufficient for the existence of a* GBRD(v,3tg,2;EA(g)).

Proof: The designs PGBRD(v,3,g,2;EA(g)), v belonging to {6,8,12}, exist by Corollary 2.3. We observe that EA(g) = Z_3 x EA(h), where h is odd. The existence of a GBRD(5,3,3,;Z_3)(Seberry (1982)) and a PGBRD(6,3,h,2;EA(h)) guarantees, by Theorem 2.13, the existence of a PGBRD(10,3,g,2;EA(g)). Similarly, the existence of a GBRD(8,4,3;Z_3)(de Launey and Seberry (1984)) and a PGBRD(8,3,h,2;EA(h)) establishes the existence of a PGBRD(16,3,g,2;EA(g)). By application of Lemma 2.7 and Theorem 2.8 we conclude that a PGBRD(v,3,g,2;EA(g)) exists for v belonging to {6,8,...}. By taking t copies of these designs we may construct a PGBRD(v,3,$\lambda = tg$,2;EA(g)) where $g \equiv 3 \pmod 6$ and v belongs to {6,8,...}.

5. Existence of PGBRD over EA(g), g even.

Theorem 5.1. *When g is even, the necessary conditions (3.7), (3.8), (3.9) and (3.10) are sufficient for the existence of a* PGBRD(v,3,tg,2;EA(g)).

Proof: When g is even, the necessary conditions (3.7), (3.8), (3.9) and (3.10) reduce to:

$tg \equiv 0 \pmod 6$, $v = 6,8,...$

$tg \equiv 2$ or $4 \pmod 6$, $v \geq 6$ and $v \equiv 0$ or $2 \pmod 6$.

By Corollary 2.5, we see these conditions are sufficient for the existence of a PGBRD(v,3,tg,2;EA(g)) when g is even.

6. Main Result

Theorem 6.1. *For the abelian group* EA(g) *a* PGBRD($v,3,\lambda,2$;EA(g)) *exists if and only if*

$$\lambda \equiv 0 \pmod g$$
$$v \geq 6$$
$$v \equiv 0 \pmod 2$$
$$\lambda v(v-2) \equiv 0 \pmod 3$$

Proof: The result follows by invoking the theorems as indicated below:

g	Theorem
$\equiv$ 1 or 5 (mod 6)	Theorem 4.1
$\equiv$ 3 (mod 6)	Theorem 4.2
$\equiv$ 0 (mod 2)	Theorem 5.1

7. Applications

Let EA(g)= $\{e=h_1,\ldots,h_g\}$ where EA(g) is the abelian group defined in section 1. Suppose that EA(g) is represented by the gxg permutation matrices $P_1,\ldots,P_g$ so that h_i corresponds to P_i, $1 \leq i \leq g$. As in Street and Rodger (1980) and Seberry (1982), we construct, by replacing each group element of a PGBRD($v,b,r,3,\lambda_1=0,\lambda_2=\lambda,2$;EA($g$)) by its corresponding gxg permutation matrix, the incidence matrix of group divisible design with $v/2$ groups each of size $2g$ and with the parameters

$$v^*=vg,\ b^*=bg,\ r^*=r,\ k^*=3,\ \lambda_1{}^*=0,\ \lambda_2{}^*=(1/g)$$

Hence we have part of Hanani's theorem but by a different approach:

Theorem 7.1.

$$\begin{aligned} \lambda &\equiv 0 \pmod g \\ v &\equiv 0 \pmod 2 \\ v &\geq 6 \\ \lambda v(v-2) &\equiv 0 \pmod 3 \end{aligned}$$

are sufficient for the existence of a group divisible design with v/2 groups each of size 2g and with the parameters:

$$v^*=vg,\ b^*=bg,\ r^*=r,\ k^*=3,\ \lambda_1{}^*=0,\ \lambda_2{}^*=\lambda/g$$

8. Acknowledgement

The author wishes to thank Professor Seberry for some helpful discussions in the preparation of this paper. The author likewise expresses his thanks to the referee whose suggestions enabled some of the proofs to be simplified.

References

Berman, G (1977). Weighing matrices and group divisible designs determined by $EG(t,p^n), t>2$. *Utilitas Math.* 12, 183-292.

Berman, G (1978). Families of generalized weighing matrices. *Canad. J. Math.* 30, 1016-1028.

Bhaskar Rao M. (1966). Group divisible family of PBIB designs. *J. Indian Statist Assoc.* 4, 14-28.

Bhaskar Rao M. (1970). Balanced orthogonal designs and their application in the construction of some BIB and group divisible designs. *Sankhyā Ser.* A 32, 439-448.

Brock, B.W. (1988). Hermitian congruence and the existence and completion of generalized Hadamard matrices. *Journal of Combinatorial theory, Series* A 49, 233-261.

Butson, A.T. (1962). Generalized Hadamard matrices. *Proc. Amer. Math. Soc.* 13, 894-898.

Butson, A.T. (1963). Relations among generalized Hadamard matrices, relative difference sets and maximal length recurring sequences. *Canad. J.Math.* 15, 42-48.

Clatworthy, W.H. (1973). *Tables of two-associate-class Partially Balanced Designs,* NBS Applied Math. Ser. No. (63).

Curran, D.J. and Vanstone, S.A. (1989). Doubly resolvable designs from generalized Bhaskar Rao designs. *Discrete Math* 73, 49-63.

Dawson, J.E. (1985). A construction for the generalized Hadamard matrices GH(4q,EA(q)) *J. Statist. Plann. and Inference* 11, 103-110.

de Launey, W. (1984). On the non-existence of generalized Hadamard matrices. *J. Statist. Plann. and Inference* 10, 385-396.

de Launey, W. (1986). A survey of generalized Hadamard matrices and difference matrices $D(k,\lambda;G)$ with large k. *Utilitas Math.* 38, 5-29.

de Launey, W. (1987A). On difference matrices, transversal orthogonal F-squares. *J. Statist. Plann. and Inference* 16, 107-135.

de Launey, W. (1987B). *(O,G)-Designs and Applications,* Ph.D. Thesis, The University of Sydney.

de Launey, W. (1989A). Square GBRDs over non-abelian groups. *Ars Combinatoria* 27, 40-49.

de Launey, W. (1989B). Some new constructions for difference matrices, generalized Hadamard matrices and balanced generalized weighing matrices. *Graphs and Combinatorics* 5, 125-135.

de Launey, W. (198). Circulant $GH(p^2;\mathbb{Z}_p)$ exist for all primes p (Preprint).

de Launey, W. and Seberry, J. (1984). Generalized Bhaskar Rao designs of block size four. *Congressus Numerantium* 41, 229-294

de Launey, W., Sarvate, D.G., Seberry, J. (1985). Generalized Bhaskar Rao designs with block size 3 over Z_4. *Ars Combinatoria* 19A, 273-286.

Delsarte, P. and Goethals, J.M. (1969). Tri-weight codes and generalized Hadamard matrices. *Information and Control* 15, 196-206.

Drake, D.A. (1979). Partial λ-geometries and generalized matrices over groups. *Canad. J. Math.* 31, 617-627.

Gibbons, P.B. and Mathon, R. (1987A). Construction methods for Bhaskar Rao and related designs. *J. Australian Math. Soc. Ser.* A 42, 5-30.

Gibbons, P.B. and Mathon, R. (1987B). Group signings of symmetric balanced incomplete block designs. *Ars Combinatoria* 23A, 123-134.

Hall, Marshall Jr. (1967) *Combinatorial Mathematics,* Waltham, Mass.

Hanani, H. (1961). The existence and construction of balanced incomplete block designs. *Ann. Math. Stat.* 32, 361-386.

Hanani, H. (1975). Balanced incomplete block designs and related designs. *Discrete Math.* 11, 255-369.
Jungnickel, D. (1979). On difference matrices, resolvable TD's and generalized Hadamard matrices. *Math. Z.* 167, 49-60.
Lam, C. and Seberry, J. (1984). Generalized Bhaskar Rao designs. *J. Statist. Plann. and Inference* 10, 83-95.
Mackenzie, C. and Seberry, J. (1988). Maximal q-ary Codes and Plotkin's bound. *Ars Combinatoria* 26B, 37-50.
Palmer, W.D. and Seberry, J. (1988). Bhaskar Rao designs over small groups. *Ars Combinatoria* 26A. 125-148.
Raghavarao, D. (1971). *Construction and combinatorial problems in design of experiments*, Wiley, New York.
Rajkundlia, D. (1983). Some techniques for constructing infinite families of BIBD's. *Discrete Math.* 44, 61-96.
Seberry, J. (1979). Some remarks on generalized Hadamard matrices and theorems of Rajkundlia on SBIBDs. *Combinatorial Mathematics IV, Lecture Notes in Math., 748,* Springer, Berlin, 154-164.
Seberry, J. (1980). A construction for generalized Hadamard matrices. *J. Statist. Plann. and Inference* 4, 365-368.
Seberry, J. (1982). Some families of partially balanced incomplete block designs. In: *Combinatorial Mathematics IX, Lecture Notes in Mathematics No. 952,* Springer-Verlag, Berlin-Heidelberg-New York, 378-386.
Seberry, J. (1984). Regular group divisible designs and Bhaskar Rao designs with block size 3. *J. Statist. Plann. and Inference* 10, 69-82.
Seberry, J. (1985). Generalized Bhaskar Rao designs of blocks size three. *J. Statist. Plann. and Inference* 11, 373-379.
Seberry, J. (1988). Bhaskar Rao designs of block size 3 over groups of order 8, Department of Computer Science, University College, The University of N.S.W., Australian Defence Force Academy, *Technical Report CC 88/4*.

Shrikhande, S.S. (1964). Generalized Hadamard matrices and orthogonal arrays of strength 2. *Canad. J. Math.* 16, 736-748.

Singh, s.J. (1982). Some Bhaskar Rao designs and applications for $k = 3$, $\lambda = 2$. *University of Indore J. Science* 7, 8-15.

Sinha, K. (1978). Partially balanced incomplete block designs and partially balanced weighing designs, *Ars Combinatoria* 6, 91-96.

Street, D.J. (1979). Generalized Hadamard matrices, orthogonal arrays and F-squares. *Ars Combinatoria* 8, 131-141.

Street, D.J. (1981). Bhaskar Rao designs from cyclotomy, *J. Austral. Math. Soc.* 29(A), 425-430.

Street, D.J. and Rodger, C.A. (1980). Some results on Bhaskar Rao designs. *Combinatorial Mathematics, VII,* 238-245, *Lecture Notes in Math., 829,* Springer, Berlin.

Vyas R. (1982). Some Bhaskar Rao designs and applications for $k = 3$, $\lambda = 4$. *University of Indore J. Science* 7, 16-25.

Wilson, R.M. (1974). A few more squares. *Proceedings of the fifth Southeastern Conference on Combinatorics, Graph Theory and Combinatorics and Computing Congressus Numerantium X.* Utilitas Math., Winnipeg, 675-680.