HOW TO CONSTRUCT A BLOCK DESIGN WITH BLOCK SIZE FOUR ADMITTING A BLOCKING SET

C. C. Lindner

Department of Algebra, Combinatorics and Analysis
Division of Mathematics
120 Mathematics Annex
Auburn University
Auburn, Alabama 36849-5307
U.S.A.

1. Introduction. Since this is a survey paper on blocking sets in block designs, the definition of a block design seems an appropriate beginning. So, here goes. A block design of index λ is a pair (P, B), where P is a finite set (the elements of which are sometimes called points) and B is a collection of subsets of P (all of the same size k) called blocks such that every pair of distinct elements of P belongs to exactly λ blocks of B. The number $|P| = n$ is called the order of the block design (P, B) and, of course, the number of blocks is $|B| = \lambda n(n-1)/k(k-1)$.

Now let (P, B) be a block design. The subset X of P is called a blocking set if and only if for each block $b \in B$, $b \cap X \neq \emptyset$ and $b \cap (P \setminus X) \neq \emptyset$. (The set X also defines a 2-colouring of (P, B) with the property that none of the blocks in B receive a monochromatic colouring. However, in what follows we will stick with calling X a blocking set rather than a 2-colouring.) The most widely studied classes of block designs are triple systems; i.e., block designs with block size 3. (See [9], for example.) Hence we begin

---

Research supported by NSF grant DMS-8703642 and NSA grant MDA-904-89-H-2016.

with a discussion of blocking sets in triple systems. As we shall see, a very short discussion!

2. <u>Triple Systems.</u> We begin with a few examples. Four to be exact.

<u>Example 2.1.</u>

(1) The unique triple system (P_1, B_1) of order 3 and index $\lambda = 1$,

$$P_1 = \{1, 2, 3\} \text{ and } B_1 = \{\{1, 2, 3\}\}.$$

Blocking sets: $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$.

(2) The unique triple system (P_2, B_2) of order 4 and index $\lambda = 2$,

$P_2 = \{1, 2, 3, 4\}$ and $B_2 = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$.

Blocking sets: $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, and $\{3, 4\}$.

(3) The unique triple system of order 3 and index $\lambda \geq 1 = \lambda$ copies of (P_1, B_1) and admits the same blocking sets as (P_1, B_1).

(4) The unique triple system of order 4 and index $\lambda = 2k \geq 2 = k$ copies of (P_2, B_2) and admits the same blocking sets as (P_2, B_2).

Now it is more or less well-known (see [8], for example) that the spectrum for triple systems (= the set of all orders for which a triple system exists) is precisely the set of all

(i) $n \equiv 1$ or $3 \pmod 6$ for $\lambda \equiv 1$ or $5 \pmod 6$,

(ii) $n \equiv 0$ or $1 \pmod 3$ for $\lambda \equiv 2$ or $4 \pmod 6$,

(iii) $n \equiv 1 \pmod 2$ for $\lambda \equiv 3 \pmod 6$, and

(iv) all $n \geq 3$ for $\lambda \equiv 0 \pmod 6$.

Unfortunately, out of all of these triple systems, only the triple systems listed in Example 2.1 admit a blocking set. This is quite easy to see. Let (P, B) be a triple system of order n and index λ admitting the blocking set X. If $|X| = x$, then $\lambda x(n-x)/2 = |B| = \lambda n(n-1)/6$ implies $n = 3$ and $x = 1$ or 2 (Example 2.1 (3)) or $n = 4$ (and hence $\lambda = 2k$) and $x = 2$ (Example 2.1 (4)).

So much for triple systems! We now irreversibly turn our attention to block designs with block size 4.

3. Block designs. In the sixties H. Hanani (see [1] for a unified discussion) proved that the spectrum for block designs with block size 4 is the set of all (i) $n \equiv 1$ or 4 (mod 12) for $\lambda \equiv 1$ or 5 (mod 6), (ii) $n \equiv 1$ (mod 3) for $\lambda \equiv 2$ or 4 (mod 6), (iii) $n \equiv 0$ or 1 (mod 4) for $\lambda \equiv 3$ (mod 6), and (iv) all $n \geq 4$ for $\lambda \equiv 0$ (mod 6).

As with triple systems, we begin with some examples. From now on, "block design" without additional quantification means block design with block size four.

Example 3.1. The following examples are examples of block designs with $\lambda = 1$.

$n = 4$ ($\lambda = 1$).

1 2 3 4 Blocking set {1, 2}

$n = 13$ ($\lambda = 1$, the projective plane of order 3).

1	2	4	10	6	7	9	2	11	12	1	7
2	3	5	11	7	8	10	3	12	13	2	8
3	4	6	12	8	9	11	4	13	1	3	9
4	5	7	13	9	10	12	5				
5	6	8	1	10	11	13	6				

Blocking set {1, 2, 3, 4, 5, 6, 7}

n = 16 (λ = 1, the affine plane of order 4).

1	2	3	16	2	6	12	14
5	7	10	14	3	4	10	15
6	8	11	15	10	11	12	16
4	9	12	13	1	4	8	14
4	5	6	16	2	5	9	15
2	8	10	13	3	6	7	13
3	9	11	14	13	14	15	16
1	7	12	15	2	4	7	11
7	8	9	16	3	5	8	12
1	5	11	13	1	6	9	10

Blocking set {1, 2, 3, 4, 5, 8, 11, 15}

n = 25 (λ = 1, [4] Design #9).

1	2	3	25	2	6	10	19	3	9	13	14	5	14	18	19	9	12	15	25
1	4	16	24	2	7	20	21	3	10	12	22	6	7	8	24	10	14	16	21
1	5	12	21	2	8	13	15	4	5	6	25	6	9	11	21	11	15	17	19
1	6	13	22	2	9	16	18	4	7	12	19	6	12	14	17	12	13	18	20
1	7	14	15	2	11	12	24	4	8	9	22	6	15	16	20	13	19	23	24
1	8	17	18	3	4	11	20	4	10	15	18	7	10	13	25	14	20	22	24
1	9	19	20	3	5	15	24	4	13	17	21	7	11	18	22	15	21	22	23
1	10	11	23	3	6	18	23	5	7	9	23	8	11	14	25	16	19	22	25
2	4	14	23	3	7	16	17	5	8	10	20	8	12	16	23	17	20	23	25
2	5	17	22	3	8	19	21	5	11	13	16	9	10	17	24	18	21	24	25

Blocking set {1, 2, 3, 4, 5, 6, 13, 14, 15, 16, 17, 18}.

n = 25 (λ = 1, [4] Design #8).

1 2 4 25	1 3 8 11	1 5 9 24	1 6 7 15	1 10 13 17
1 12 19 21	1 14 18 22	1 16 20 23	2 3 14 19	2 5 8 16
2 6 9 12	2 7 18 23	2 10 11 22	2 13 15 20	2 17 21 24
3 4 10 15	3 5 7 13	3 6 21 25	3 9 16 22	3 12 17 23
3 18 20 24	4 5 11 20	4 6 8 14	4 7 16 21	4 9 17 18
4 12 13 24	4 19 22 23	5 6 17 22	5 10 21 23	5 12 14 15
5 18 19 25	6 10 19 20	6 11 23 24	6 13 16 18	7 8 19 24
7 9 10 25	7 11 14 17	7 12 20 22	8 9 15 23	8 10 12 18
8 13 21 22	8 17 20 25	9 11 13 19	9 14 20 21	10 14 16 24
11 12 16 25	11 15 18 21	13 14 23 25	15 16 17 19	15 22 24 25

Oops! No blocking sets.

The above examples illustrate two things. One is that there are non-trivial block designs which admit blocking sets, and the other is that there is no use in attempting to show that every block design admits a blocking set, since it's not true. (Design #8 is a counter-example.) Since blocking sets in block designs (for any λ) cannot be ruled out by a cardinality argument a la triple systems, and not every block design admits a blocking set (it is easy to construct infinite classes of block designs which fail to admit blocking sets, just take direct products) the following question is the only reasonable question we can ask: Can we construct (= does there exist) a block design admitting a blocking set for every admissible order and index?

Quite recently, a complete solution of this problem (modulo a handful of possible exceptions) was obtained by the author, D. G. Hoffman, and K. T. Phelps [2, 3]. In this brief survey, we synthesize these results leaving out the gory details. The interested reader can consult the original papers for a complete account. After all, this is a survey!

In view of the remarks at the beginning of this section, we break the constructions into four parts: $\lambda \equiv 1$ or 5 (mod 6), $\lambda \equiv 2$ or 4 (mod 6), $\lambda \equiv 3$ (mod 6), and $\lambda \equiv 0$ (mod 6).

In what follows we will refer to a block design with index λ as a "λ-fold block design". When $\lambda = 1$, this will be shortened to simply "block design".

4. $\underline{\lambda \equiv 1 \text{ or } 5 \pmod 6}$. Since the spectrum for λ-fold block designs is the set of all $n \equiv 1$ or 4 (mod 12) for $\lambda \equiv 1$ or 5 (mod 6), any solution for $\lambda = 1$ is also a solution for $\lambda \geq 5$ and $\lambda \equiv 1$ or 5 (mod 6). Just list each block of a block design (admitting a blocking set) λ times. The blocking set is the same. In this set of notes we concentrate on $\lambda = 1$ exclusively. Hence any possible exception for $\lambda = 1$ runs through everything. However, there are only three possible exceptions for $\lambda = 1$, and we eliminate two of these for $\lambda \equiv 1$ or 5 (mod 6) ≥ 5 in Section 8. So it's not exactly the end of the world!

To make sure that everyone is on the same wave length we state the following well-known definitions.

A pairwise balanced design (PBD) of index λ is a pair (P, B), where P is a finite set and B is a collection of subsets (called blocks), not necessarily of the same size, such that every pair of distinct elements of P belongs to exactly λ blocks of B.

A group divisible design (GDD) of index λ is a triple (X, G, B) where G is a collection of subsets (called groups) which partition X and B is a collection of subsets (called blocks) such that

$(X, \lambda G \cup B)$ is a PBD of index λ. (λG = each group of G is listed λ times.) As is the usual custom we will abbreviate "PBD or GDD of index $\lambda = 1$" to simply "PBD or GDD".

Example 4.1. (X, G, T) is a GDD design of order $|X| = 24$, group size 6, and block size 3.

$$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24\};$$

$$G = \begin{cases} \{1, 2, 3, 4, 5, 6\}, \\ \{7, 8, 9, 10, 11, 12\}, \\ \{13, 14, 15, 16, 17, 18\}, \text{ and} \\ \{19, 20, 21, 22, 23, 24\}. \end{cases}$$

T =

1	8	13	1	16	23	4	9	16	6	15	22
1	7	20	11	15	23	4	10	19	8	16	22
1	14	19	2	11	16	4	15	20	5	10	13
7	13	19	2	12	23	10	16	20	5	9	24
2	7	14	2	15	24	3	12	13	5	14	23
2	8	19	12	16	24	3	11	22	8	13	23
2	13	20	3	8	17	3	14	21	6	9	14
8	14	20	3	7	24	11	13	21	6	10	23
1	10	17	3	18	23	4	11	14	6	13	24
1	9	22	7	17	23	4	12	21	10	14	24
1	18	21	4	7	18	4	13	22	5	12	17
9	17	21	4	8	23	12	14	22	5	11	20
2	9	18	4	17	24	5	8	15	5	18	19
2	10	21	8	18	24	5	7	22	11	17	19
2	17	22	3	10	15	5	16	21	6	11	18
10	18	22	3	9	20	7	15	21	6	12	19
1	12	15	3	16	19	6	7	16	6	17	20
1	11	24	9	15	19	6	8	21	12	18	20

Let (X, G, T) be a GDD with block size 3. The mapping $\alpha : T \to X$ is called a <u>nesting</u> if and only if (X, G, T^*) is a GDD with block size 4 and index $\lambda = 2$, where $T^* = \{\{a, b, c, t\alpha\} \mid t = \{a, b, c\} \in T\}$.

<u>Example 4.2.</u> A nesting α of the GDD (X, G, T) in Example 4.1.

T^* =

	t		tα		t		tα		t		tα		t		tα
1	8	13	20	1	16	23	12	4	9	16	19	6	15	22	7
1	7	20	14	11	15	23	1	4	10	19	15	8	16	22	6
1	14	19	8	2	11	16	23	4	15	20	9	5	10	13	24
7	13	19	1	2	12	23	15	10	16	20	4	5	9	24	14
2	7	14	19	2	15	24	11	3	12	13	22	5	14	23	10
2	8	19	13	12	16	24	2	3	11	22	14	9	13	23	5
2	13	20	7	3	8	17	24	3	14	21	12	6	9	14	23
8	14	20	2	3	7	24	18	11	13	21	3	6	10	23	13
1	10	17	22	3	18	23	8	4	11	14	21	6	13	24	9
1	9	22	18	7	17	23	3	4	12	21	13	10	14	24	6
1	18	21	10	4	7	18	23	4	13	22	11	5	12	17	20
9	17	21	1	4	8	23	17	12	14	22	4	5	11	20	18
2	9	18	21	4	17	24	7	5	8	15	22	5	18	19	12
2	10	21	17	8	18	24	4	5	7	22	16	11	17	19	5
2	17	22	9	3	10	15	20	5	16	21	8	6	11	18	19
10	18	22	2	3	9	20	16	7	15	21	5	6	12	19	17
1	12	15	24	3	16	19	10	6	7	16	21	6	17	20	11
1	11	24	16	9	15	19	3	6	8	21	15	12	18	20	6

(X, G, T^*) is a GDD with group size 6, block size 4, and index $\lambda = 2$.

<u>Theorem 4.3</u> (Doug Stinson [7]). There exists a GDD (X, G, T) with group size 6 and block size 3 which can be nested of every order $|X| = 6k$, except $k = 1, 2, 3$, and 6. □

<u>The $12k + 1$ Construction.</u> Let (X, G, T) be a GDD with group size 6 and block size 3. Let α be a nesting of (X, G, T). Let $P = \{\infty\} \cup (X \times \{1, 2\})$ and define a collection of blocks B as follows:

(1) For each $g \in G$ let $(\{\infty\} \cup (g \times \{1, 2\}), g^*)$ be a block design of order 13 with blocking set $g \times \{1\}$ (Example 3.1) and place the blocks of g^* in B, and

(2) if x and y belong to different groups place the 2 blocks $\{(x, 1), (y, 1), (z, 1), (t\alpha, 2)\}$ and $\{(x, 2), (y, 2), (z, 2), (t\alpha, 1)\}$ in B, where $t = \{x, y, z\} \in T$.

Then (P, B) is a block design of order $12k + 1$ and $X \times \{1\}$ is a blocking set. □

<u>The $12k + 4$ Construction.</u> In the $12k + 1$ Construction set $P = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (X \times \{1, 2\})$ and replace (1) by: For each $g \in G$ let $(\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (g \times \{1, 2\}), g^*)$ be a block design of order 16 such that $\{\infty_1, \infty_2, \infty_3, \infty_4\} \in g^*$ and $\{\infty_1, \infty_2\} \cup (g \times \{1\})$ is a blocking set (Example 3.1) and place the blocks of g^* in B.

Then (P, B) is a block design of order $12k + 4$ and $\{\infty_1, \infty_2\} \cup (X \times \{1\})$ is a blocking set. □

<u>Theorem 4.4.</u> There exists a block design admitting a blocking set of <u>every</u> order $n \equiv 1$ or $4 \pmod{12}$, except possibly $n = 37, 40$, and 73.

Proof. Example 3.1, Theorem 4.3, and the $12k + 1$ and $12k + 4$ Constructions take care of everything except 28, 37, 40, 73, and 76. Ad hoc constructions (see [2]) handle the cases $n = 28$ and 76 leaving only $n = 37$, 40, and 73 as possible exceptions. $\square$

Remark. The author doesn't believe for a moment that the possible exceptions listed in the above theorem are really exceptions. (Neither does Alex Rosa.) It remains only for someone to produce the required block designs.

5. $\lambda \equiv 2$ or 4 (mod 6). The spectrum for λ-fold block designs is the set of all $n \equiv 1, 4, 7$, or 10 (mod 12) for $\lambda \equiv 2$ or 4 (mod 6). As with $\lambda \equiv 1$ or 5 (mod 6), we construct only 2-fold block designs admitting a blocking set, since for $\lambda \geq 4$ and $\lambda \equiv 2$ or 4 (mod 6) we can just take repeated copies of a 2-fold block design (admitting a blocking set). This, of course, stretches any possible exceptions for $\lambda = 2$ through all $\lambda \equiv 2$ or 4 (mod 6). Since there are only 5 possible exceptions for $\lambda = 2$, it's not something to lie awake at night worrying over. If $n \equiv 1$ or 4 (mod 12), except for $n = 37$, 40, and 73, we can double the blocks of a block design admitting a blocking set. Hence, other than these three cases, we need look only at the construction of 2-fold block designs of order $n \equiv 7$ or 10 (mod 12) which admit a blocking set. We begin with an example.

Example 5.1. A 2-fold block design of order 7.

1	2	4	7
2	3	5	1
3	4	6	2
4	5	7	3
5	6	1	4
6	7	2	5
7	1	3	6

Blocking set $\{1, 2, 3\}$.

Theorem 5.2 (C. C. Lindner and C. A. Rodger [5]). There exists a GDD (X, G, T) with group size 3 and block size 3 which can be nested of every order $|X| = 6k+3 > 15$. $\square$

The $12k + 7$ Construction. Let (X, G, T) be a GDD with group size 3 and block size 3. Let α be a nesting of (X, G, T). Set $P = \{\infty\} \cup (X \times \{1, 2\})$ and define a collection of blocks B as follows:

(1) For each $g \in G$ let $(\{\infty\} \cup (g \times \{1, 2\}), g^*)$ be a 2-fold block design of order 7 with blocking set $g \times \{1\}$ and place the blocks of g^* in B, and

(2) if x and y belong to different groups place the 4 blocks $\{(x, 1), (y, 1), (z, 1), (t\alpha, 2)\}$, $\{(x, 1), (y, 1), (z, 1), (t\alpha, 2)\}$, $\{(x, 2), (y, 2), (z, 2), (t\alpha, 1)\}$, and $\{(x, 2), (y, 2), (z, 2), (t\alpha, 1)\}$ in B, where $t = \{x, y, z\} \in T$.

Then (P, B) is a 2-fold block design of order $12k + 7$ and $X \times \{1\}$ is a blocking set. $\square$

Lemma 5.3. There exists a 2-fold block design admitting a blocking set of every order $n \equiv 7 \pmod{12}$, except possibly $n = 19$.

Proof. Write $12k+7 = 2(6k+3) + 1$ and use the $12k+7$ Construction. □

More examples!

Example 5.4. (2-fold block designs).

$n = 10$.

1	2	8	10
2	3	9	6
3	4	10	7
4	5	6	8
5	1	7	9
1	3	4	6
2	4	5	7
3	5	1	8
4	1	2	9
5	2	3	10
1	6	7	10
2	7	8	6
3	8	9	7
4	9	10	8
5	10	6	9

Blocking set $\{1, 2, 3, 4, 5\}$.

n = 22.

16	7	8	9	19	3	8	12	22	2	7	12	18	3	5	15	21	10	12	13
16	2	6	10	19	2	5	13	22	6	8	15	18	9	10	14	22	1	5	9
16	4	5	11	19	1	7	14	22	11	13	14	19	4	6	9	22	3	4	10
16	1	3	13	19	10	11	15	16	7	8	9	19	3	8	12	22	2	7	12
16	12	14	15	20	5	7	10	16	2	6	10	19	2	5	13	22	6	8	15
17	1	8	10	20	4	8	13	16	4	5	11	19	1	7	14	22	11	13	14
17	3	7	11	20	3	6	14	16	1	3	13	19	10	11	15	16	17	19	22
17	5	6	12	20	1	2	15	16	12	14	15	20	5	7	10	17	18	20	16
17	2	4	14	20	9	11	12	17	1	8	10	20	4	8	13	18	19	21	17
17	9	13	15	21	2	3	9	17	3	7	11	20	3	6	14	19	20	22	18
18	2	8	11	21	1	6	11	17	5	6	12	20	1	2	15	20	21	16	19
18	1	4	12	21	5	8	14	17	2	4	14	20	9	11	12	21	22	17	20
18	6	7	13	21	4	7	15	17	9	13	15	21	2	3	9	22	16	18	21
18	3	5	15	21	10	12	13	18	2	8	11	21	1	6	11				
18	9	10	14	22	1	5	9	18	1	4	12	21	5	8	14				
19	4	6	9	22	3	4	10	18	6	7	13	21	4	7	15				

Blocking set {1, 2, 3, 4, 9, 12, 13, 14, 19, 20, 22}.

Let (X, o) be a quasigroup and $H = \{h_1, h_2, h_3, \ldots, h_m\}$ a partition of X. The subsets h_i belonging to H are called holes. If for each hole $h_i \in H$, (h_i, o) is a subquasigroup of (X, o), then (X, o) is called a quasigroup with holes H.

Lemma 5.5. There exists a pair of orthogonal quasigroups of order $4t + x$ with t holes of size 4 and one hole of size x, for all $x \in \{1, 3\}$ and $t \notin \{1, 2, 3, 6, 10\}$.

<u>Proof.</u> Let (X, G, B) be a GDD of order $|X| = 5t$ with group size t and block size 5 (equivalent to 3 mutually orthogonal quasigroups of order t). Truncate one of the groups to size $x = 1$ or 3 and denote by (X^*, G^*, B^*) the derived GDD. Any one of the deleted points gives t disjoint blocks of size 4 which we can take to be holes along with the truncated group of size x. Placing a pair of orthogonal quasigroups on each hole and a pair of orthogonal idempotent quasigroups on the remaining groups and blocks completes the construction. □

<u>The $12k + 10$ Construction.</u> Write $12k + 10 = 6(4t + x) + 4$, where $x = 1$ or 3. Let (X, o_1) and (X, o_2) be a pair of orthogonal quasigroups of order $4t + x$ with holes $H = \{h_1, h_2, h_3, \ldots, h_m\}$, where $|h_1| = x$ and $|h_2| = |h_3| = \ldots = |h_m| = 4$. Let $P = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (X \times \{1, 2, 3, 4, 5, 6\})$ and define a collection of blocks B as follows:

(1) Let $(\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (h_1 \times \{1, 2, 3, 4, 5, 6\}), h_1^*)$ be a 2-fold block design of order 10 or 22 (depending on whether $|h_1| = x = 1$ or 3, Example 5.4) with blocking set $\{\infty_1, \infty_2\} \cup (h_1 \times \{1, 2, 3\})$ and place the blocks of h_1^* in B,

(2) for h_i, $i = 2, 3, \ldots, m$, let $(\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (h_i \times \{1, 2, 3, 4, 5, 6\}), h_i^*)$ be a 2-fold block design of order 28 containing the blocks $\{\infty_1, \infty_2, \infty_3, \infty_4\}$ and $\{\infty_1, \infty_2, \infty_3, \infty_4\}$ with blocking set $\{\infty_1, \infty_2\} \cup (h_i \times \{1, 2, 3\})$ and place the blocks of $h_i^* \setminus \{\{\infty_1, \infty_2, \infty_3, \infty_4\}, \{\infty_1, \infty_2, \infty_3, \infty_4\}\}$ in B, and

(3) for each ordered pair (x, y), x and y in different holes of H, and for each $(a, a, b, c) \in D = \{(1, 1, 2, 5), (2, 2, 3, 6), (3, 3, 4, 1), (4, 4, 5, 2), (5, 5, 6, 3), (6, 6, 1, 4)\}$ place the block $\{(x, a), (y, a), (x \circ_1 y, b), (x \circ_2 y, c)\}$ in B.

As with the previous constructions it is routine to see that (P, B) is a 2-fold block design of order $12k + 10$ and that $\{\infty_1, \infty_2\} \cup (X \times \{1, 2, 3\})$ is a blocking set. □

Lemma 5.6. There exists a 2-fold block design admitting a blocking set of every order $n \equiv 10 \pmod{12}$, except possibly $n = 34$, 46, 58.

Proof. Example 5.4, Lemma 5.5, and the 12k+10 Construction takes care of everything except for $n = 34$, 46, 58, 70, 82, 94, 154, 166, 250, and 262. The cases $n = 70$, 82, 94, 154, 166, 250, and 262 are handled in [2] using ad hoc constructions (which we omit here since this is a survey paper) leaving only $n = 34$, 46, and 58 as possible exceptions. □

Theorem 5.7. There exists a 2-fold block design admitting a blocking set of every order $n \equiv 1 \pmod{3}$, except possibly $n = 19$, 34, 37, 46, and 58.

Proof. Theorem 4.4, Lemma 5.3, and Lemma 5.6 guarantee the existence of a 2-fold block design of every order $n \equiv 1 \pmod{3}$, except possibly $n = 19$, 34, 40, 46, 58, and 73. Although $n = 40$ and 73 are possible exceptions for $\lambda = 1$, they can be removed as possible exceptions for $\lambda = 2$. We refer the reader to [2] for the appropriate details. □

Remarks. As with the case $\lambda = 1$, the author has no doubt that the possible exceptions in Theorem 5.7 are purely a figment of a fertile imagination.

6. $\lambda \equiv 3 \pmod 6$. The spectrum for λ-fold block designs for $\lambda \equiv 3 \pmod 6$ is the set of all $n \equiv 0$ or $1 \pmod 4$. As with the conditions $\lambda \equiv 1$ or $5 \pmod 6$ and $\lambda \equiv 2$ or $4 \pmod 6$, we construct only 3-fold block designs admitting a blocking set. Since there are no exceptions for $\lambda = 3$, this gives a complete solution. How about that!

As usual, some examples.

Example 6.1. (3-fold block designs).

n = 4.

1	2	3	4
1	2	3	4
1	2	3	4

Blocking set $\{1, 2\}$

n = 5.

1	2	3	4
1	2	3	5
1	2	4	5
1	3	4	5
2	3	4	5

Blocking set $\{1, 2\}$

n = 8.

1	2	4	8	3	5	6	7
2	3	5	8	1	4	6	7
3	4	6	8	1	2	5	7
4	5	7	8	1	2	3	6
5	6	1	8	2	3	4	7
6	7	2	8	1	3	4	5
7	1	3	8	2	4	5	6

Blocking set $\{1, 2, 3, 4\}$

n = 9.

1	2	3	4	2	4	6	8
1	2	8	9	2	4	7	9
1	3	8	9	2	5	7	8
1	4	5	6	2	5	6	9
1	4	5	7	4	5	8	9
1	2	3	5	3	5	7	9
1	6	7	8	3	5	6	8
1	6	7	9	3	4	6	9
2	3	6	7	3	4	7	8

Blocking set $\{1, 2, 3, 9\}$

n = 12.

(i, j), (1+i, j), (i, 1+j), (2+i, 1+j)
(i, j), (3+i, j), (i, 1+j), (3+i, 1+j)
(1+i, j), (2+i, j), (3+i, j), (i, 1+j)
(1+i, j), (2+i, j), (4+i, j), (3+i, 1+j)
$i \in Z_6$ (mod 6) and $j \in Z_2$ (mod 2)

Blocking set $\{(i, 0) \mid i \in Z_6\}$

n = 17.

(i, j), (4+i, j), (i, 1 + j), (4+i, 1+j)
(i, j), (2+i, j), (5+i, j), (3+i, 1+j)
(i, j), (1+i, j), (2+i, j), (3+i, 1+j)
(i, j), (1+i, j), (4+i, j), (1+i, 1+j)
∞ , (i, j), (2+i, j), (4+i, 1+j)
$i \in Z_8$ (mod 8) and $j \in Z_2$ (mod 2)

Blocking set $\{(i, 0) \mid i \in Z_8\}$

$n = 24$.

(i, j), $(1+i, j)$, $(6+i, j)$, $(2+i, 1+j)$
$(1+i, j)$, $(2+i, j)$, $(4+i, j)$, $(i, 1+j)$
$(1+i, j)$, $(2+i, j)$, $(5+i, j)$, $(10+i, 1+j)$
$(1+i, j)$, $(4+i, j)$, $(8+i, j)$, $(10+i, 1+j)$
$(1+i, j)$, $(3+i, j)$, $(5+i, j)$, $(i, 1+j)$
(i, j), $(6+i, j)$, $(i, 1+j)$, $(6+i, 1+j)$
(i, j), $(5+i, j)$, $(i, 1+j)$, $(5+i, 1+j)$
$i \in Z_{12}$ (mod 12) and $j \in Z_2$ (mod 2)

Blocking set $\{(i, 0) \mid i \in Z_{12}\}$.

We now generalize the definition of nesting. Let (X, G, T) be a GDD with block size 3 and index $\lambda = 3$. The mapping $\alpha: T \to X$ is called a nesting if and only if (X, G, T^*) is a GDD with block size 4 and index $\lambda = 6$, where $T^* = \{\{a, b, c, t\alpha\} \mid t = \{a, b, c\} \in T\}$.

Example 6.2. (X, G, T) is a GDD of order $|X| = 10$, group size 2, block size 3, and index $\lambda = 3$ and α is a nesting.

$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,

$G = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}\}$, and

$T(T^*) =$

	t		tα		t		tα		t		tα		t		tα
1	6	10	4	3	1	9	8	6	7	10	2	8	4	9	6
1	8	10	6	3	6	8	9	6	2	8	4	8	2	5	10
1	3	5	8	4	5	9	2	6	3	9	7	9	4	8	2
1	4	7	9	4	1	8	5	6	1	4	10	9	5	7	4
2	6	10	3	4	2	10	7	7	4	4	1	9	1	3	6
2	7	9	5	4	6	7	10	7	2	9	3	9	2	6	8
2	4	5	7	5	8	9	1	7	3	10	5	10	3	7	1
2	3	8	10	5	1	7	3	7	1	5	9	10	5	8	3
3	5	10	1	5	4	10	8	8	3	6	2	10	2	4	5
3	2	7	6	5	2	3	9	8	1	10	4	10	1	6	7

(X, G, T^*) is a GDD with group size 2, block size 4, and index $\lambda = 6$.

__Theorem 6.3__ (C. C. Lindner, C. A. Rodger, and D. R. Stinson [6]). There exists a GDD (X, G, T) with group size 2 or 4, block size 3, and index $\lambda = 3$ which can be nested of every order $|X| = 2k$, $k \notin \{2, 4, 6, 8, 12\}$. □

__The 4k Construction.__ Let (X, G, T) be a GDD of order $2k$ with group size 2 or 4, block size 3, and index $\lambda = 3$. Further, let α be a nesting of (X, G, T). Let $P = X \times \{1, 2\}$ and define a collection of blocks B as follows:

(1) For each $g \in G$ let $(g \times \{1, 2\}, g^*)$ be a 3-fold block design (of order 4 or 8, Example 6.1) with blocking set $g \times \{1\}$, and

(2) for each triple $t = \{x, y, z\} \in T$ place each of the blocks $\{(x, 1), (y, 1), (z, 1), (t\alpha, 2)\}$ and $\{(x, 2), (y, 2), (z, 2), (t\alpha, 1)\}$ in B.

Then (P, B) is a 3-fold block design of order $4k$ and $X \times \{1\}$ is a blocking set. □

__The 4k + 1 Construction.__ Exactly the same as the 4k Construction __but__ with $P = \{\infty\} \cup (X \times \{1, 2\})$ and (1) __replaced__ by: For each $g \in G$ let $(\{\infty\} \cup (g \times \{1, 2\}), g^*)$ be a 3-fold block design (of order 5 or 9, Example 6.1) with blocking set $g \times \{1\}$.

Then (P, B) is a 3-fold block design of order $4k + 1$ and $X \times \{1\}$ is a blocking set. □

__Theorem 6.4.__ The spectrum for 3-fold block designs admitting a blocking set is __precisely__ the set of all $n \equiv 0$ or $1 \pmod 4$.

__Proof.__ Theorem 6.3 and the $4k$ and $4k+1$ Constructions produce a 3-fold block design admitting a blocking set of every order $n \notin \{12,$

13, 16, 17, 24, 25}. The cases $n = 12, 17$, and 24 are taken care of by Example 6.1, and $n = 13, 16$, and 25 by Example 3.1 (triple each block). □

7. $\lambda \equiv 0 \pmod 6$. The spectrum for λ-fold block designs for $\lambda \equiv 0 \pmod 6$ is the set of all $n \geq 4$. We construct only 6-fold block designs admitting a blocking set. It turns out that (like the case $\lambda \equiv 3 \pmod 6$) there are no exceptions for $\lambda = 6$ and so the solution for $\lambda = 6$ gives a complete solution. In view of Theorem 6.4 we need consider only the cases $n \equiv 2$ or $3 \pmod 4$ for 6-fold block designs. (For $n \equiv 0$ or $1 \pmod 4$ take two copies of a 3-fold block design.) As always, we begin with some examples.

Example 7.1.

$n = 6$. All 4-element subsets of $\{1, 2, 3, 4, 5, 6\}$. Blocking set $\{1, 2, 3\}$.

$n = 11$.

$(1+i, j), (2+i, j), (4+i, j), (1+i, 1+j)$ 2 times
$(1+i, j), (2+i, j), (3+i, j), (i, 1+j)$ $\infty, (1+i, j), (4+i, j), (i, 1+j)$ $\infty, (2+i, j), (3+i, j), (i, 1+j)$ $(2+i, j), (3+i, j), (2+i, 1+j), (3+i, 1+j)$
$i \in Z_5 \pmod 6$ and $j \in Z_2 \pmod 2$

Blocking set $\{(i, 0) \mid i \in Z_5\}$

$\underline{n = 14}$. Let (Z_{13}, B) be a 1-fold block design with blocking set $\{0, 1, 2, 6, 9, 11\}$ (Example 3.1).

B 5 times
∞, 1+i, 2+i, 5+i ∞, 1+i, 3+i, 8+i
$i \in Z_{13}$ (mod 13)

Blocking set $\{\infty, 0, 1, 2, 6, 9, 11\}$

$\underline{n = 15}$.

(1+i, j), (2+i, j), (4+i, j), (i, 1+j) (5 times)
∞ , (i, j) , (2+i, j), (i, 1+j) ∞ , (i, j) , (3+i, j), (i, 1+j) (i, j) , (1+i, j), (1+i, 1+j), (i, 1+j)
$i \in Z_7$ (mod 7) and $j \in Z_2$ (mod 2)

Blocking set $\{(i, 0) \mid i \in Z_7\}$.

$\underline{n = 18}$.

(1+i, j), (2+i, j), (3+i, j), (5+i, 1+j) (1+i, j), (3+i, j), (5+i, j), (i, 1+j) (1+i, j), (2+i, j), (6+i, j), (i, 1+j) (1+i, j), (2+i, j), (4+i, j), (5+i, 1+j) (1+i, j), (3+i, j), (7+i, j), (i, 1+j) (2+i, j), (6+i, j), (5+i, j), (i, 1+j) (i, j), (3+i, j), (6+i, j), (1+i, 1+j) (i, j), (1+i, j), (i, 1+j), (1+i, 1+j) (i, j), (2+i, j), (i, 1+j), (2+i, 1+j) (i, j), (4+i, j), (i, 1+j), (4+i, 1+j)
$i \in Z_9$ (mod 9) and $j \in Z_2$ (mod 2)

Blocking set $\{(i, 0) \mid i \in Z_9\}$.

$n = 19$.

$(1+i, j), (2+i, j), (4+i, j), (i, 1+j)$ 5 times
$(1+i, j), (2+i, j), (4+i, j), (7+i, 1+j)$ $\infty, (i, j), (2+i, j), (i, 1+j)$ $\infty, (i, j), (4+i, j), (i, 1+j)$ $(i, j), (1+i, j), (5+i, j), (2+i, 1+j)$ $(i, j), (1+i, j), (5+i, j), (4+i, 1+j)$ $(i, j), (2+i, j), (5+i, j), (3+i, 1+j)$ $(i, j), (3+i, j), (i, 1+j), (3+i, 1+j)$
$i \in Z_9 \pmod 9$ and $j \in Z_2 \pmod 2$

Blocking set $\{(i, 0) \mid i \in Z_9\}$

It should now come as no surprise that the main constructions for 6-fold block designs admitting a blocking set use GDDs which can be nested. Hence the following (by now repetitive) definition. Let (X, G, T) be a GDD with block size 3 and index $\lambda = 6$. The mapping $\alpha : T \to X$ is called a nesting if and only if (X, G, T^*) is a GDD with block size 4 and index $\lambda = 12$, where $T^* = \{\{a, b, c, t\alpha\} \mid t = \{a, b, c\} \in T\}$.

Example 7.2. (X, G, T) is a GDD of order 11 with group sizes 2 and 3, block size 3, and index $\lambda = 6$, which can be nested.

$X = Z_{13}$, $G = \{\{0, 4\}, \{1, 5\}, \{2, 6\}, \{3, 7\}, \{8, 9, 10\}\}$, and $B = \{\{10, i, 1+i; 2+i\}, \{i, 10, 1+i; 3+i\}, \{i, 1+i, 10; 2+i\}, \{i, 2+i, 5+i; 10\}, \{9, i, 2+i; 5+i\}, \{i, 9, 2+i; 7+i\}, \{i, 3+i, 9; 1+i\}, \{i, 5+i, 3+i; 9\}, \{8, i, 5+i; 3+i\}, \{i, 8, 7+i; 6+i\}, \{i, 6+i, 8; 5+i\}, \{i, 7+i, 6+i; 8\} \mid i \in Z_8\}$. Set $T = \{\{a, b, c\} \mid \{a, b, c; d\} \in B\}$ and define α by $\{a, b, c\}\alpha = d$ if and only if $\{a, b, c; d\} \in B$. Then (X, G, T) is a GDD of order 11 with block size 3, $\lambda = 6$, and group size 2 or 3 and α is a nesting.

Theorem 7.3. (C. C. Lindner, D. G. Hoffman and K. T. Phelps [3]). There exists a GDD (X, G, T) of order $|X| = 2k+1$ with block size 3, $\lambda = 6$, and $2k+1 > |g| > 2$ for all $g \in G$ which can be nested for all $k > 5$. □

We now proceed to the main constructions for $\lambda = 6$.

The $4k + 2$ Construction. Let (X, G, T) be a GDD of order $2k + 1$ with block size 3, $\lambda = 6$, and such that $2 \cdot |g|$ belongs to the known spectrum of 6-fold block designs admitting a blocking set of size $|g|$. Further let α be a nesting of (X, G, T). Let $P = X \times \{1, 2\}$ and define a collection of blocks B as follows:

(1) For each $g \in G$ let $(g \times \{1, 2\}, g^*)$ be a 6-fold block design with blocking set $g \times \{1\}$, and

(2) for each triple $t = \{x, y, z\} \in T$ place each of the blocks $\{(x, 1), (y, 1), (z, 1), (t\alpha, 2)\}$ and $\{(x, 2), (y, 2), (z, 2), (t\alpha, 1)\}$ in B.

Then (P, B) is a 6-fold block design of order $4k + 2$ and $X \times \{1\}$ is a blocking set. □

The $4k + 3$ Construction. Exactly the same as the $4k + 2$ Construction but with $P = \{\infty\} \cup (X \times \{1, 2\})$ and (1) replaced by: For each $g \in G$ let $(\{\infty\} \cup (g \times \{1, 2\}), g^*)$ be a 6-fold block design with blocking set $g \times \{1\}$.

Then (P, B) is a 6-fold block design of order $4k + 3$ and $X \times \{1\}$ is a blocking set. □

Theorem 7.4. The spectrum for 6-fold block designs admitting a blocking set is precisely the set of all $n > 4$.

Proof. Because of Theorem 6.4 we need consider only the cases $n \equiv 2$ or 3 (mod 4). We begin by taking care of the cases $n = 6, 7, 10, 11, 14, 15, 18$, and 19. The cases $n = 6, 11, 14, 15, 18$, and 19 are taken care of by Example 7.1 and $n = 7$ and 10 by Examples 5.1 and 5.4 (triple each block). We can now assume $n \equiv 2$ or 3 (mod 4) > 22 and that we have constructed 6-fold block designs of every order $2m + i < n$, $i = 0$ or 1, with a blocking set of size m. Write $n = 2(2k + 1)$ or $2(2k + 1) + 1$. Since $n > 22$, $2k + 1 > 11$. Hence by Theorem 7.3 there exists a GDD (X, G, T) of order $2k + 1$, block size 3, $\lambda = 6$, and $|g| > 2$ for all $g \in G$ which can be nested. Since $|g| < 2k+1$, $2 \cdot |g| < n$ and so there exists a 6-fold block design of order $2 \cdot |g|$ with a blocking set of order $|g|$. Applying the $4k + 2$ or $4k + 3$ Construction completes the proof. □

8. Concluding remarks. Combining all of the results in this paper we have the following theorem.

Theorem 8.1. There exists a λ-fold block design admitting a blocking set for every admissible (n, λ) except possibly for $(n \in \{37, 40, 73\}, \lambda = 1)$, $(n = 37, \lambda \equiv 1$ or 5 (mod 6) $\geq 5)$, and $(n \in \{19, 34, 37, 46, 58\}, \lambda \equiv 2$ or 4 (mod 6)$)$.

Proof. The elimination of 40 and 73 for $\lambda \equiv 1$ or 5 (mod 6) ≥ 5 is achieved by pasting together 2-fold and 3-fold block designs of orders 40 and 73. □

Open problem. Nobody in the city of Auburn believes for a moment that the possible exceptions listed in the statement of Theorem 8.1 are really exceptions at all. The elimination of these possible exceptions is no doubt a tractable problem. However, after struggling with blocking sets in block designs for over a year the song "But not for me" is running through my head.

References

1. H. Hanani, Balanced incomplete block designs and related designs, Discrete Math., 11 (1975), 255-369.

2. D. G. Hoffman, C. C. Lindner, and K. T. Phelps, Blocking sets in designs with block size four, (to appear).

3. D. G. Hoffman, C. C. Lindner, and K. T. Phelps, Blocking sets in designs with block size four II, (to appear).

4. E. S. Kramer, S. S. Magliveras, and R. Mathon, The Steiner systems S(2, 4, 25) with nontrivial automorphism group, Annals of Discrete Math., (to appear).

5. C. C. Lindner and C. A. Rodger, The spectrum of nested Steiner triple systems of order 3 (mod 6), Ars Combinatoria, 23 (1987), 75-80.

6. C. C. Lindner, C. A. Rodger, and D. R. Stinson, Nesting of cycle systems of odd length, Annals of Discrete Math., (to appear).

7. D. R. Stinson, The spectrum of nested Steiner triple systems, Graphs and Combinatorics, 1 (1985), 189-191.

8. A. P. Street and D. J. Street, "Combinatorics of Experimental Design", Clarendon Press, Oxford, 1987, pp. 400.

9. Topics on Steiner systems, Annals of Discrete Mathematics, 7 (1980), eds. C. C. Lindner and A. Rosa.